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THERMAL FLUCTUATIONS IN THE GROSS-NEVEU MODEL WITH $U(1)$-SYMMETRY AT SMALL N

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The chiral Gross-Neveu model is one of the most popular toy models for QCD. In the past, it has been studied in detail in the large-N limit. In this paper we study its small-N behavior at finite temperature in 2+1 dimensions. We show that at small N the phase diagram of this model is *principally* different from its behavior at $N \to \infty$. For a small number N of fermions, the model possesses two characteristic temperatures T_{KT} and T^*. That is, at small N, along with a quasiordered phase $0 < T < T_{KT}$ the system possesses a very large region of precursor fluctuations $T_{KT} < T < T^*$ which disappear only at a temperature T^*, substantially higher than the temperature T_{KT} of the Kosterlitz-Thouless transition.

In this contribution we discuss the small-N behavior of the Gross-Neveu (GN) [1] model with $U(1)$-symmetry in $2+1$ dimensions at finite temperature. The Gross-Neveu model is a field theoretic model of zero-mass fermions with quartic interaction, which provides us with considerable insight into the mechanisms of spontaneous symmetry breakdown and is considered to be an illuminating toy model for QCD. Our small-N study is motivated by recent results in the theory of superconductivity in the regimes, where BCS mean-field theory is not valid.

In the past years, remarkable progress was made in the theory of superconductivity in understanding mechanisms of symmetry breakdown in the regimes of strong interaction and low carrier density. It was observed that, in general, a Fermi system with attraction possesses two distinct characteristic temperatures corresponding to pair formation and pair condensation. That is, in a strong-coupling superconductor Cooper pairs are formed at a certain temperature T^* although there is no macroscopic occupation of zero momentum

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level and thus no phase coherence and no symmetry breakdown. The temperature should be lowered to $T_c(\ll T^*)$ in order to make these pairs condense and establish phase coherence. The large region $T_c < T < T^*$ where there exist Cooper pairs but no continuous symmetry is broken is called *pseudogap phase* (see Ref. [2] and numerous references therein). Thus the symmetry breakdown in a strong-coupling superconductor resembles onset of long-range order in ^{4}He, where one can also introduce formally a characteristic temperature of thermal decomposition of a He atom. However it does not mean that this temperature is connected in any respect with the temperature of the onset of phase coherence. In fact, the BCS scenario, where the superconductive phase transition can approximately be described as a pair-formation transition, is very exceptional, since there is only one characteristic temperature T_c. That is, the BCS scenario holds true only at infinitesimally weak coupling strength or very high carrier density.

Since the original work by Bardeen, Cooper, and Schrieffer [3] in 1957, the BCS theory has been a source of inspiration in particle physics. In particular, it had direct influence on the construction of the Nambu-Jona-Lasinio and Gross-Neveu models [4,5]. In this spirit, the pseudogap concept was recently introduced to particle physics in Ref. [6], sparking recently many intensive discussions.

Let us now return to the Gross-Neveu model. In this article we would like to readdress the pseudogap properties of the Gross-Neveu model, extending the initial work of Ref. [6] to finite temperatures and higher dimensions. At finite temperatures in the limit of large N its behavior closely resembles a BCS superconductor [7]. We show that a very rich physical structure, similar to the phase diagram of a strong-coupling superconductor, emerges in the small-N limit in the chiral GN model.

The chiral GN model [1] has the following Lagrange density [8]

$$\mathcal{L} = \bar{\psi}_a i \partial\!\!\!/ \, \psi_a + \frac{g_0}{2N}\left[\left(\bar{\psi}_a \psi_a\right)^2 + \left(\bar{\psi}_a i\gamma_5 \psi_a\right)^2\right],$$

where the index a runs from 1 to N. The fields $\psi(x)$ can be integrated out yielding the collective field action (for a detailed discussion see Ref. [9]):

$$\mathcal{A}_{\rm coll}[\sigma,\pi] = N\left\{-\frac{\sigma^2+\pi^2}{2g_0} - i\mathrm{Tr}\,\log\left[i\partial\!\!\!/ - \sigma(x) - i\gamma_5\pi\right]\right\}.$$

This model is invariant under the continuous set of chiral $O(2)$ transformations which rotate σ and π fields into each other. In the large-N limit, the

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model has a second-order phase transition at which fermions acquire mass. At zero temperature in 2+1 dimensions it is accompanied by an appearance of a massless composite Goldstone boson (for details see e.g. Ref. [9]). In the symmetry-broken phase the model is characterized by a typical "mexican hat" effective potential. The propagator of the massive σ fluctuations can be readily extracted and it coincides with the σ-propagator of the ordinary GN model [7,9]:

$$G_{\sigma'\sigma'} = -\frac{i}{N}\left[g_0^{-1} - i\ \mathrm{tr}\int \frac{d^3k}{(2\pi)^3}\frac{(\not{k}+M)(\not{k}-\not{q}+M)}{(k^2-M^2)[(k-q)^2-M^2]}\right]^{-1},$$

where M is the mass dynamically acquired by fermions. According to standard dimensional reduction arguments, the system is at finite temperature effectively two-dimensional and thus the Coleman theorem forbids the spontaneous breakdown of the $U(1)$-symmetry. However, as it was shown by Witten [10], this does not preclude the system from generating a fermion mass. As it was shown in Ref. [10] by employing "modulus-phase" variables

$$\sigma + i\pi = |\Delta|e^{i\theta} \tag{1}$$

one can see that the system generates the fermion mass $M = |\Delta|$ that coincides with the modulus of the complex order parameter, but its phase remains incoherent and the correlators of the complex order parameter have algebraic decay. Existence of the *local* gap modulus Δ does not contradict the Coleman theorem since Δ is neutral under $U(1)$ transformations. Thus at low temperature in 2+1 dimensions there appears an "almost" Goldstone boson that becomes a real Goldstone boson at exactly zero temperature.

Let us here first study the effective potential of the model at finite temperature in the leading-order approximation and then take into account the next-to-leading-order corrections. Following Ref. [10], the fermion mass at finite temperature is given by the gap equation which coincides with the gap equation for the ordinary GN model with discrete symmetry (for detailed calculations see Refs. [7,6,9]):

$$1 = g_0\,\mathrm{tr}\,(1)\int \frac{d^2k}{(2\pi)^2}\frac{1}{2E}\tanh\left(\frac{E}{2T}\right), \tag{2}$$

where E stands for $\sqrt{k^2+\Delta^2}$. In the leading-order mean-field approximation we have a situation similar to the BCS superconductor: There is a gap that disappears at a certain temperature which we denote by T^* in the sequel. It

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can be expressed via the gap function at zero temperature:

$$T^* = \frac{\Delta(0)}{2\log 2}. \tag{3}$$

Near T^* the gap function has the following behavior in the mean-field approximation:

$$\Delta(T) = T^* 4\sqrt{\log 2}\sqrt{1 - \frac{T}{T^*}}. \tag{4}$$

On the other hand, at low temperatures the gap function receives an exponentially small temperature correction:

$$\Delta(T) = \Delta(0) - 2T\exp\left[-\frac{\Delta(0)}{T}\right]. \tag{5}$$

Let us note once more that a straightforward calculation of next-to-leading-order corrections leads to the conclusion that the gap should be exactly zero at any finite temperature in 2+1 dimensions. However, as shown in Ref. [10], such a direct calculation of corrections misses the essential physics of a two-dimensional problem. The fluctuations can be made arbitrarily weak by decreasing temperature in 2+1 dimensions (or e.g. increasing N in 2D zero temperature calculations in Ref. [10]) and then the system possesses a very well-defined "mexican hat" effective potential that determines the fermion mass. Due to phase fluctuations in the degenerate valley of the potential, the average of the complex gap function is zero, however there exists a gap locally (i.e. in some sense the system in its low energy domain degenerates to a nonlinear sigma model). In 2+1 dimensions, as the temperature approaches zero the thermal fluctuations in the degenerate valley of the effective potential gradually vanish and at $T = 0$ a local gap becomes a real gap. The most interesting effect happens however when temperature is increased. It was anticipated before that there is only one characteristic temperature in such a system, namely the temperature of the Kosterlitz-Thouless (KT) transition which coincides with the temperature of the formation of the local gap. This scenario holds true only for a very large number of field components. In general, the model has two characteristic temperatures like in the case of a superconductor with a pseudogap. In order to show this we have to go beyond the mean-field approximation.

Let us make an expansion around a saddle-point solution and derive the propagator of the imaginary part of the field Δ that has a pole at $q^2 = 0$.

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The procedure is standard and will not be reproduced here (for details see e.g. Refs. [6,9]):

$$G_{\Delta'_{\rm im}\Delta'_{\rm im}} = \frac{1}{N}\left[\frac{1}{8\pi\Delta(T)}\tanh\left(\frac{\Delta(T)}{2T}\right)\right]^{-1}\frac{i}{q^2}. \tag{6}$$

The propagator (6) characterizes the stiffness of the phase fluctuations in the degenerate minimum of the effective potential. It gives the following expression for the kinetic term of phase fluctuations for the chiral GN model:

$$E_{\rm kin} = \int d^2x \frac{N}{8\pi}\Delta(T)\tanh\left(\frac{\Delta(T)}{2T}\right)[\nabla\theta]^2. \tag{7}$$

Now we have all the tools to find the position of the KT transition in the chiral GN model. It is well known that the KT transition cannot be found by straightforward perturbative methods. In order to find the KT transition point one should first observe that the system is described by a complex scalar field. The key feature is that the field θ is cyclic: $\theta = \theta + 2\pi n$. In two dimensions such a system possesses excitations of the form of vortices and antivortices that are attracted to each other by a Coulomb potential. At low temperatures, the vortices and antivortices form bound pairs. The grand-canonical ensemble of the pairs exhibits quasi-long-range correlations. At a certain temperature T_{KT}, the vortex pairs break up, which is the Kosterlitz-Thouless phase-disorder transition [11,12]. This transition was studied in detail in the field theory of a pure phase field $\theta(x)$, with a Hamiltonian

$$\mathcal{H} = \frac{\beta}{2}[\partial\theta(x)]^2, \tag{8}$$

where β is the stiffness of the phase fluctuations. In our case the coefficient β depends on the temperature and on the parameters of the GN models, namely the number of field components and Δ [see Eq. (7)].

The temperature of the Kosterlitz-Thouless phase transition in the system (8) is given by Ref. [11,12]:

$$T_{\rm KT} = \frac{\pi}{2}\beta. \tag{9}$$

In order to study the phase-disorder transition in the chiral GN model, we should solve a *set* of equations, namely the equation for $T_{KT}(\Delta, N)$ that follows from the kinetic term, and Eq. (2) for the gap modulus $\Delta(T_{KT}, N)$ that follows from the effective potential. Thus in our case the phase transition

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is a competition of two processes, the thermal excitation of directional fluctuations in the degenerate valley of the effective potential and the thermal depletion of the stiffness coefficient.

Let us first consider the case of small N. From expressions (3), (5), (7), and (9) we see that, in the regime of small N, $T_{KT} \ll T^*$. In this regime the temperature corrections to the phase stiffness are exponentially suppressed. Thus, at temperatures $T \ll T^*$, the asymptotic expression for the kinetic term (7) reads

$$H_{\rm kin} = \int d^2x \frac{N}{8\pi} \Delta(0) [\nabla\theta]^2, \tag{10}$$

and the Kosterlitz-Thouless transition will take place at the temperature

$$T_{KT} = \frac{N}{8} \Delta(0). \tag{11}$$

This is significantly lower at small N than the temperature (3) at which the gap modulus disappears. For the ratio T_{KT}/T^* at small N we obtain:

$$\frac{T_{KT}}{T^*} = \frac{N \log(2)}{4}. \tag{12}$$

So with decreasing N, the separation of T_{KT} and T^* increases. Let us now turn to the regime where N is no longer small. Here we see from Eqs. (3), (4), (7), and (9) that T_{KT} tends to T^* from below. The Hamiltonian (7) reads in this limit near T^*:

$$H_{\rm kin} = \int dx \frac{N}{16\pi} \frac{\Delta(T)^2}{T} [\nabla\theta]^2. \tag{13}$$

From Eqs. (3), (4), (13), and (9) we find the following expression for the behavior of T_{KT} at large N:

$$T_{KT} \simeq T^* \left(1 - \frac{1}{N \log(2)}\right). \tag{14}$$

This equation explicitly shows a merging of the temperatures T_{KT} and T^* in the limit of large N. This can be interpreted as the restoration of the "BCS-like" scenario for the quasi-condensation in the limit $N \to \infty$. The ratio T_{KT}/T^* is displayed in Fig. 1.

Thus the phase diagram of the model at *small* N consists of the following phases at non-zero temperature:

(i) $0 < T < T_{KT}$: the low temperature quasi-ordered phase with bound

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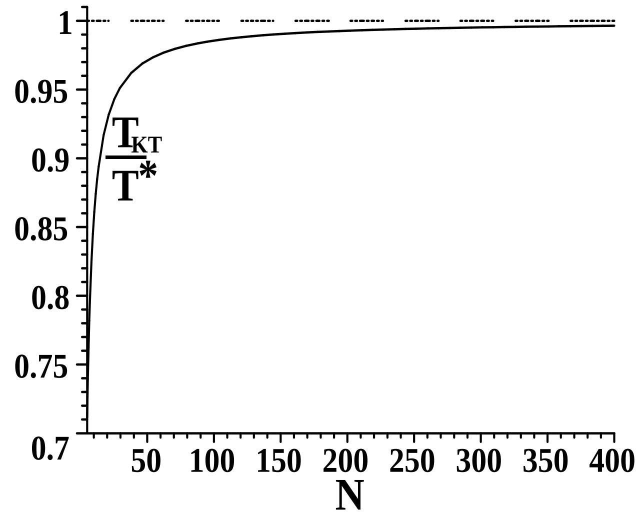

Figure 1. Recovery of a "BCS-like" scenario for quasicondensation at large N in the chiral GN model. The solid curve is the ratio of the temperature of the KT transition (T_{KT}) and the characteristic temperature of the formation of the effective potential (T^*). This ratio tends from below to unity (the horizontal dashed line) as N is increasing and the region of precursor fluctuations shrinks.

vortex-antivortex pairs,
(ii) $T_{KT} < T < T^*$: the phase analogous to the *pseudogap phase* of superconductors, i.e. the chirally symmetric phase with unbound vortex-antivortex pairs that exhibit violent precursor fluctuations and a nonzero local modulus of the complex gap function,
(iii) $T > T^*$: high temperature "normal" chirally symmetric phase.

The mechanism of the phase separation is very transparent with the key feature being the fact that the stiffness is proportional to N [see Eq. (7)]. At large N, the directional fluctuations are energetically extremely expensive and thus, the phase transition is controlled basically by the modulus of the order parameter. On the other hand, the stiffness is low at small N, and the thermal excitation of the fluctuations in the degenerate valley of the effective potential starts governing the phase transition in the system.

Now let us briefly discuss the physical meaning of T^* and what is expected to happen when the system reaches it at small N. At first, we can conclude from simple physical reasoning in analogy with superconductivity that the

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appearance of the second characteristic temperature is very natural. Besides the fact that the phase analogous to the intermediate phase $T_{KT} < T < T^*$ is the dominating region on a phase diagram of strong-coupling and low carrier density superconductors, similar effects are known in a large variety of condensed matter systems such as excitonic condensates, Josephson junction arrays and many other systems. One of the most illuminating examples of the appearance of the pseudogap phase is the chiral GN model in $2+\epsilon$ dimensions at zero temperature where this phenomenon is governed by quantum dynamical fluctuations at small N [6]. In $D = 2+\epsilon$ the presence of two small parameters in the problem has allowed to prove the existence and the different physical origin of two characteristic values of a renormalized coupling constant and of the formation of an intermediate pseudogap phase [6]. We can also observe that the mean-field approximation gives a second-order phase transition at T^* which is certainly an artifact since much above T_{KT} there are violent thermal phase fluctuations. These fluctuations should wash out the phase transition at T^* which should degenerate to a smeared crossover as it happens in superconductors. Apparently, this crossover cannot be studied adequately in the framework of an $1/N$-expansion. The best insight into the properties of the system in the region $T_{KT} < T < T^*$ can be obtained by numerical simulations. Although the KT transition is very hard to observe in simulations, the hint for the phase separation would be a gradual degradation of the transition at T^* with decreasing N.

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References

[1] D. Gross and A. Neveu, *Phys. Rev. D* **10**, 3235 (1974).
[2] E. Babaev and H. Kleinert, *Phys. Rev. B* **59**, 12083 (1999).
[3] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
[4] Y. Nambu and G. Jona Lasinio, *Phys. Rev.* **122**, 345 (1961); **124**, 246 (1961); V.G. Vaks and A.I. Larkin, *JETP (Sov. Phys.)* **13**, 979 (1961); A.A. Anselm, *JETP (Sov. Phys.)* **9**, 608 (1959).
[5] H. Kleinert, *On the Hadronization of Quark Theories*, Lectures presented

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

at the Erice Summer Institute 1976, in *Understanding the Fundamental Constituents of Matter*, Ed. A. Zichichi (Plenum Press, New York, 1978), pp. 289-390.

[6] H. Kleinert and E. Babaev, *Phys. Lett. B* **438**, 311 (1998), eprint: hep-th/9809112.

[7] B. Rosenstein, B.J. Warr, and S.H. Park, *Phys. Rep.* **205**, 59 (1991).

[8] In $2+1$ dimensions we choose γ-matrices as in the review Ref. [7]: $\gamma^\mu = \sigma^\mu \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $i\gamma_5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

[9] H. Kleinert, *Particles and Quantum Fields*, Lecture notes at FU Berlin.

[10] E. Witten, *Nucl. Phys. B* **145**, 110 (1978).

[11] V.L. Berezinskii, *Zh. Eksp. Teor. Fiz.* **59**, 907 (1970); J. Kosterlitz and D. Thouless, *J. Phys. C* **6**, 1181 (1973).

[12] H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. I: *Superflow and Vortex Lines* and Vol. II: *Stresses and Defects* (World Scientific, Singapore, 1989).