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A REMARK ON THE NUMERICAL VALIDATION OF TRIVIALITY FOR SCALAR FIELD THEORIES BY HIGH-TEMPERATURE EXPANSIONS

P. BUTERA AND M. COMI

Istituto Nazionale di Fisica Nucleare,
Dipartimento di Fisica, Università di Milano-Bicocca,
3 Piazza della Scienza, 20126 Milano, Italy
E-mails: butera@mi.infn.it; comi@mi.infn.it

We suggest a simple modification of the usual procedures of analyzing the high-temperature (strong-coupling or hopping-parameter) expansions of the renormalized four-point coupling constant in the $\vec{\phi}^4_4$ lattice scalar field theory. As a result we can more convincingly validate numerically the triviality of the continuum limit taken from the high-temperature phase.

There has been a steady accumulation of suggestive numerical and analytical evidence, but not yet a complete rigorous proof that the continuum limit of the lattice regularized N-component $\vec{\phi}^4_4$-theory describes a free (or "trivial") field theory [1–4]. The basic clues of this paradoxical no-interaction property were indicated almost half a century ago [5], but more stringent studies of this conjecture had to wait for the developments of the Renormalization Group (RG) theory [6]. The modern rigorous analyses of Refs. [1,2,7–10] have finally proved that the $\vec{\phi}^4_d$-theory is non-trivial [11] in $d \leq 3$ and trivial in $d \geq 5$ dimensions, at least for not too large values of N. Triviality is expected to occur also in $d = 4$ dimensions. Since, however, the rigorous results in this direction are still partial, various routes to recover an interesting continuum theory have also been explored [12,13].

The lattice Euclidean $\vec{\phi}^4_4$-theory with $O(N)$ symmetry is defined by the action [14]

$$S = \sum_x \Big\{ -\beta \sum_\mu \vec{\phi}_x \cdot \vec{\phi}_{x+\mu} + \vec{\phi}_x^2 + \lambda(\vec{\phi}_x^2 - 1)^2 \Big\}, \tag{1}$$

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where $\vec{\phi}_x$ is a real N-component field at the lattice site x and μ is the unit vector in the μ direction and $\lambda \geq 0$.

For $\beta \uparrow \beta_c(N,\lambda)$, at fixed positive λ the model has a critical point where a second-order transition occurs from a high-temperature (HT) paramagnetic phase to a low-temperature ferromagnetic phase. In the $\lambda \to \infty$ limit, the model leads to the lattice nonlinear $O(N)$-symmetric σ-model or, equivalently, to the N-vector spin model.

The construction of a continuum limit of the lattice theory is reduced to the determination of its critical properties. Here we shall consider only the continuum limit taken from the HT phase. In the context of the RG theory a detailed description is obtained for the asymptotic cutoff dependence of the correlation functions in terms of the weak-coupling expansion of the theory's beta function.

If we set $\tau(N,\lambda) = 1 - \beta/\beta_c(N,\lambda)$, the perturbative RG theory yields [15] the following critical behavior as $\tau(N,\lambda) \downarrow 0$ for the correlation length

$$\xi^2(\beta,N,\lambda) = A_\xi^2(N) \frac{|\ln(\tau(N,\lambda))|^{G(N)}}{\tau(N,\lambda)} \Big[1 + O\Big(\ln(|\ln(\tau)|)/\ln(\tau)\Big)\Big], \tag{2}$$

where $G(N) = N + 2/N + 8$.

The asymptotic behavior of the susceptibility is completely similar:

$$\chi(\beta,N,\lambda) = A_\chi(N) \frac{|\ln(\tau(N,\lambda))|^{G(N)}}{\tau(N,\lambda)} \Big[1 + O\Big(\ln(|\ln(\tau)|)/\ln(\tau)\Big)\Big]. \tag{3}$$

The fourth derivative of the free energy at zero field $\chi_4(\beta,N,\lambda)$ has the behavior

$$\chi_4(\beta,N,\lambda) = A_4(N) \frac{|\ln(\tau(N,\lambda))|^{4G(N)-1}}{\tau(N,\lambda)^4} \Big[1 + O\Big(\ln(|\ln(\tau)|)/\ln(\tau)\Big)\Big]. \tag{4}$$

In terms of χ, ξ^2 and χ_4, the dimensionless renormalized 4-point coupling constant $g_r(N,\lambda)$ is defined by the critical value of the ratio

$$g_r(\beta,N,\lambda) = -\frac{\chi_4(\beta,N,\lambda)}{\xi^4(\beta,N,\lambda)\chi^2(\beta,N,\lambda)} \tag{5}$$

as $\tau(N,\lambda) \downarrow 0$.

It can be shown that $g_r(\beta,N,\lambda)$ is non-negative [16] for all β. If $g_r(\beta,N,\lambda)$ vanishes as $\tau(N,\lambda) \downarrow 0$, the continuum limit of the lattice model taken from the high-temperature phase describes a (generalized)-free-field theory [17],

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namely a theory where the connected parts of the four-point and higher-point functions vanish.

As $\tau(N,\lambda) \downarrow 0$, the perturbative RG yields the leading asymptotic behavior, with a well-specified universal amplitude [14]

$$g_r(\beta,N,\lambda) \approx \frac{c(N)}{|\ln(\tau(N,\lambda))|}\Big[1 + O\Big(\ln(|\ln(\tau)|)/\ln(\tau)\Big)\Big], \tag{6}$$

where $c(N) = 2/b_1(N)$ and $b_1(N) = N + 8/48\pi^2$ is the first non-vanishing coefficient of the beta-function.

Therefore the perturbative RG theory implies that $g_r(\beta,N,\lambda) \to 0$ as $\tau(N,\lambda) \downarrow 0$, namely that the continuum $\vec{\phi}^4_4$-model is trivial.

Since the validity of the results in Eqs. (2), (3), (4), and (6) is based upon the (unwarranted) perturbative determination of the beta function, it is interesting, at least, to try to confirm them within different approximation schemes. To this purpose, various HT or strong-coupling expansion analyses [6,14,18–22] have been performed. Many extensive Monte Carlo (MC) lattice simulations [23–31] have also been carried out and progressively refined over the years, due to the rapid evolution of computers and the improvement of algorithms and data analysis. Until now, both the stochastic simulation and the HT series studies have been generally carried out in a completely parallel way. For instance, in the case of the $4d$ self-avoiding walk model (namely, the $\vec{\phi}^4_4$-model for $N = 0$ and $\lambda \to \infty$) a HT expansion of χ up to order β^{13} on the hypercubic lattice has been analyzed [18] in order to detect the logarithmic factor predicted by the RG theory in Eq. (3) and to estimate its exponent. For the same purpose, χ and ξ^2 have been measured [23,30] in high precision MC simulations. Analogous studies [19,21] have been devoted to the $4d$ Ising model (namely, the $N = 1$ case for $\lambda \to \infty$) using series $\mathcal{O}(\beta^{17})$ for χ and χ_4 on the hypercubic lattice. The MC simulations of Refs. [24,25,28] have tried to show directly the consistency of the estimates of $g_r(\beta,N,\lambda)$ with the elusive asymptotic behavior shown in Eq. (6). A somewhat different approach, based on the scaling properties of the partition function zeroes in the complex β plane, has been adopted in the simulations of Ref. [31]. Moreover, various analytical or semianalytical approaches [32,33] have also been pursued.

All of these non-perturbative calculations have given results consistent, or at least not in contrast, with the predicted critical behaviors of χ, ξ, and g_r. However, the cited computations are somewhat limited in their extent, since only the $N = 0$ and $N = 1$ cases for $\lambda \to \infty$ and the $N = 1$ case for finite λ

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to order β^{10} have been considered until now [20]. Moreover, it is a common experience that it is difficult to uncover numerically a logarithmic behavior or, more generally, a logarithmic correction to a power behavior. Indeed, as the computations proceed deeply into the asymptotic regime, their reliability decreases and the uncertainties of their results often reach almost the same order of magnitude as the effects that have to be revealed. In the case of the HT expansions, we have also observed that the methods of Refs. [18,21], which were very effective in the $N = 0$ and $N = 1$ cases, are not as successful when $N > 1$.

In this note, we do not present new data, but reconsider the HT expansions calculated through order β^{14}, more than a decade ago, by Lüscher and Weisz [14] for χ, ξ^2, and χ_4 on the hypercubic lattice. They have produced and analyzed these series to obtain a bound on the Higgs particle mass as a consequence of the triviality of the scalar sector in the standard model. From the outset, they used also the assumption of validity of the perturbative RG and therefore avoided to place too much confidence in the HT series within the critical regime. We make no such assumption, but rather suggest a slightly different and hopefully more convincing way of analyzing the series, which takes advantage of the specific smoothness features of the HT expansion approach and, in the end, also turns out to be completely consistent with the RG results. We study how accurately an obvious consequence of Eq. (6), rather than the equation itself, is confirmed by computations.

One can see that Eq. (6) implies

$$F(\beta, N, \lambda) = \tau(N, \lambda) \frac{d}{d\beta} \left[\frac{1}{g_r(\beta, N, \lambda)} \right] = \frac{b_1(N)}{2} + O\Big(\ln\left(|\ln(\tau)|\right)/\ln^3(\tau) \Big) \tag{7}$$

as $\tau(N, \lambda) \downarrow 0$.

In order to confirm triviality, we then show at least that $F(\beta, N, \lambda)$ has a finite limit $\tilde{F}(N, \lambda)$, as $\tau(N, \lambda) \downarrow 0$. The analysis is even more compelling if i) $\tilde{F}(N, \lambda) \approx \tilde{F}(N)$, namely if the quantity $\tilde{F}(N, \lambda)$ does not depend on λ, as required by universality, and if, moreover,
ii) $2\tilde{F}(N)/b_1(N) \approx 1$, namely if, unlike in previous approaches, it is possible to show complete quantitative consistency between the strong-coupling estimate of g_r, including the universal amplitude $2/b_1(N)$, and the weak-coupling RG prediction of Eqs. (6) and (7).

Since the HT series coefficients can be written as simple rational functions of N [34], we can easily repeat the analysis on a wide range of values of N

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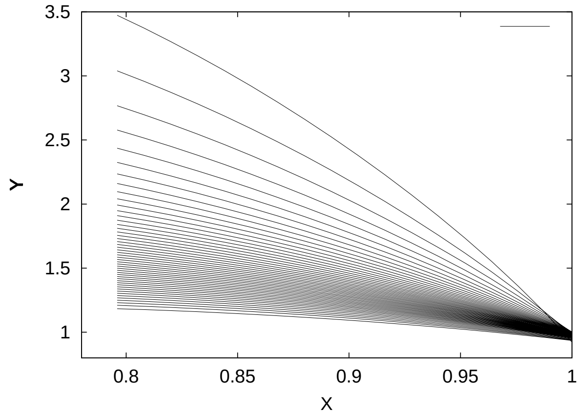

Figure 1. The quantity $y = 2\tilde{F}(\beta, N, \lambda)/b_1(N)$ versus the scaled variable $x = \beta/\beta_c(N, \lambda)$, with $N = 4$. Going from the top to the bottom of the figure, the various curves correspond to increasing values of λ between 0 and ∞.

and thus further corroborate this result.

Let us stress that the whole analysis cannot be easily performed in the context of a MC simulation, whereas it is completely straightforward in a HT series approach.

The main results of our procedure can be summarized into a couple of figures. In Fig. 1 we have plotted for $N = 1$ the quantity $y = 2\tilde{F}(\beta, N, \lambda)/b_1(N)$ versus the scaled variable $x = \beta/\beta_c(N, \lambda)$, in order to be able to compare the curves obtained for various fixed values of λ. The values of $\beta_c(N, \lambda)$ used here are estimated by an analysis of the susceptibility expansions. We have calculated $F(\beta, N, \lambda)$ by simply forming Padé approximants (PA) of its HT expansion. For each value of λ, we have plotted only the highest non-defective PA, namely the [6/6] or the [6/7] approximants, as appropriate. The other approximants of sufficiently high order have the same behavior and are not reported in the figure. As $x \to 1$, the various curves obtained in this way appear to tend to unity, independently of λ, within a good approximation, thus confirming i). We expect that the residual small spread of the limiting values would be significantly reduced if we could further reduce the uncertainties in the determination of $\beta_c(N, \lambda)$ and if we could devise approximants more accurately allowing for the slowly vanishing corrections to scaling indicated in Eq. (7). Of course, these improvements are strictly related.

We have performed the above calculation for various values of N. Figure 2

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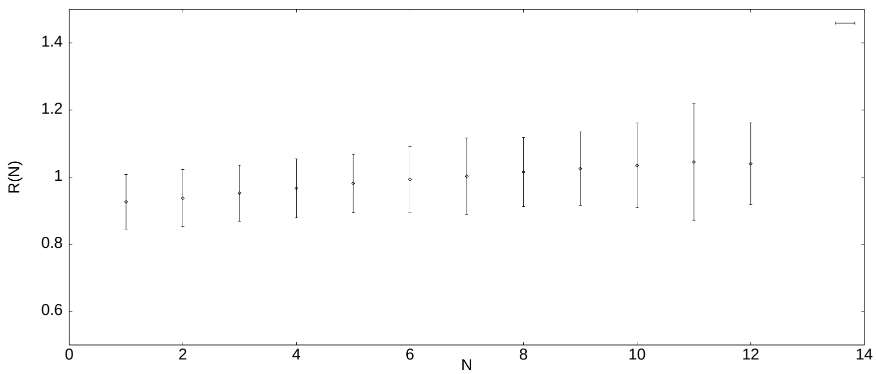

Figure 2. The ratio $R(N) = 2\tilde{F}(N)/b_1(N)$ versus N.

shows the ratio $R(N) = 2\tilde{F}(N)/b_1(N)$ versus N. For each value of N the reported error reflects the spread of this quantity. Within a fair approximation, $R(N)$ appears to be unity over a wide range of values of N, thus confirming ii).

Therefore both figures indicate a good quantitative agreement with the asymptotic formula Eq. (6), obtained within the perturbative RG approach. These very general results are unlikely to be accidental and completely confirm the conventional expectations concerning triviality.

The HT series we have used in this first test are unfortunately too short to make a more refined analysis possible. The favourable results, however, encourage to resume this study as soon as our systematic work of HT series extension by the linked-cluster method [35] will make new longer expansions for this model available. We do not expect results to be qualitatively different from this preliminary study, but significant quantitative improvements.

At this point, it is interesting to recall that a similar result on the consistency between the strong coupling and the weak coupling approaches has already been reported for the $O(N)$ symmetric lattice nonlinear σ-model in two dimensions. Indeed, also for this model, the first perturbative coefficient of the beta- and of the gamma-functions has been computed [36] with a good accuracy, starting from a strong-coupling expansion.

In conclusion, we have shown that a small modification of the current procedures of numerical analysis is sufficient to shift the emphasis from a difficult qualitative question, namely how accurately an elusive logarithmic behavior is reproduced by an approximation scheme of inevitably limited

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precision, to a more quantitative issue. Our technique of analysis for HT series is not more involved than the usual ones, while, for all values of N, it seems to produce a more convincing numerical validation of the perturbative-RG triviality predictions within the strong-coupling approach.

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