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MULTILOOP ϕ^4-THEORY AT CRITICALITY

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We review the contribution of Hagen Kleinert's group to the computation of the $\overline{\text{MS}}$ renormalization group functions of four-dimensional $O(N)$ ϕ^4-theory at *five* loops. The structure of the β-function beyond this order is also discussed from the point of view of recent developments in connecting knot theory with the transcendental numbers which appear at three and higher loops as well as the large N expansion.

1 Introduction

It is a pleasure and an honour to participate in this publication celebrating Professor Dr. Hagen Kleinert's sixtieth birthday and to review one of his many contributions related to multiloop calculations underlying critical phenomena [1]. In particular we will recall the role a particular four-dimensional scalar field theory plays in understanding phase transitions occuring in nature. For example, ϕ^4-field theory endowed with an $O(N)$ internal symmetry in three space-time dimensions underpins the statistical properties of long polymer chains, ($N = 0$), it relates to phase transitions in Ising like systems and the physics of classical fluid liquid vapour transition, ($N = 1$), it deals with Helium superfluid transition, ($N = 2$), and ferromagnetic systems, ($N = 3$) [2].

A key element to understand the physics of these various phase transitions experimentally and theoretically are the fundamental critical exponents. These are universal quantities from the point of view of the renormalization group equation and govern the scaling behavior of, say, the specific heat or susceptibility. Indeed they are presently being measured more accurately and

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hence current theoretical input must be ingenious enough to compete with the progress being made. Since the critical exponents of the underlying quantum field theory are simply related to the renormalization group functions evaluated at the critical coupling or temperature, the issue is one of computing quantities such as the β-function or field anomalous dimensions to as high a loop order as is humanly or computerly possible. There are several approaches to this problem in relation to scalar field theories such as ϕ^4-theory which is renormalizable in four dimensions but superrenormalizable in three dimensions. For instance, one can compute directly in the three-dimensional theory (see, for example, Refs. [3–5] and references therein). Alternatively, the four-dimensional theory can be renormalized to determine the renormalization group functions in the $\overline{\text{MS}}$ scheme. These are used to deduce the critical exponents as series in ϵ, where $d = 4 - 2\epsilon$ is the space-time dimension which then need to be resummed since the expressions are formally divergent or asymptotic. Indeed there are various ways of dealing with this resummation problem to improve numerical accuracy for the exponents though the standard method is Padé-Borel summation. Alternative approaches have been developed more recently by Kleinert, for example, which are based on a strong-coupling method of a variational technique which does not make use of renormalization methods and has been applied to the strictly three-dimensional model [6] and the $(4 - 2\epsilon)$-dimensional model [7–9]. Numerical results obtained for the final three-dimensional critical exponents from this new approach are impressive and also competitive with other series improvement techniques. However, the main aim of this article is to recall the computation of Kleinert *et al.* [1] and some issues concerning the renormalization group equations of scalar ϕ^4-field theories in four and other dimensions, since resummation techniques rely heavily upon having the explicit information at hand, as well as important recent insights into the structures which lie beyond the five-loop results of Ref. [1].

The paper is organised as follows. Background to ϕ^4-theory is discussed in Section 2, where the five-loop results of Ref. [1] are reviewed. In Section 3 the structure of the four-dimensional $\overline{\text{MS}}$ renormalization group functions at six and more loops is examined in relation to connections with knot theory and the large N expansion. Concluding remarks are contained in Section 4.

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2 The ϕ^4-Theory

The underlying quantum field theory which governs the above critical phenomena is ϕ^4-theory whose (massless) Lagrangian is

$$L = \frac{1}{2}\partial_\mu \phi_0^a \partial^\mu \phi_0^a + \frac{16\pi^2}{4!} g_0 (\phi_0^a \phi_0^a)^2, \tag{1}$$

where the field ϕ_0^a and the coupling constant g_0 are bare and $1 \leq a \leq N$. The factor of $16\pi^2$ associated with the coupling constant is included so that the expansion of the renormalization group functions is in terms of g rather than the usual $g/(16\pi^2)$. It is instructive to compare Eq. (1), reformulated in terms of renormalized parameters, with the criticality version. Introducing the renormalized parameters $\phi = \phi_0 \sqrt{Z_\phi}$ and $g = g_0 Z_g$, we get

$$L = \frac{Z_\phi}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \frac{16\pi^2}{4!} \tilde{\mu}^{4-d} g Z_g Z_\phi^2 (\phi^a \phi^a)^2, \tag{2}$$

where Z_ϕ and Z_g are computed in a regularized version of the theory. Since the four-dimensional models are related to lower-dimensional models, one uses here dimensional regularization where the arising singularities will appear as poles in ϵ. Moreover, they are subtracted in a (modified) minimal way which allows to compute the renormalization group functions to as high a loop order as possible. The scale $\tilde{\mu}$ is introduced in Eq. (2) to ensure the coupling constant to remain dimensionless in d dimensions.

By contrast in the critical region in d dimensions the action S governing the phase transition has a different form. If ξ is some length scale at the fixed point, then S is formally

$$S = \int d^d x \left[\frac{\xi^{-\eta}}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \frac{\xi^\chi}{2}\sigma \phi^a \phi^a - \frac{3\xi^{\chi_{\sigma^2}}\sigma^2}{32\pi^2 g} - \frac{\sigma}{2\lambda} + \ldots \right], \tag{3}$$

where the auxiliary field σ has been introduced in order to have a trivalent interaction at criticality. If one ignores the higher terms, the elimination of the trivalent interaction would restore the four-point interaction of Eq. (1). The omitted terms involve composite operators of the fields ϕ^a, σ and their derivatives. Each will have its own coupling constant which will either be relevant, irrelevant or marginal at criticality. For instance, the coupling constant λ corresponds to the coupling of the linear term in σ. If one ignores for the moment the presence of the scaling dimension with each term, then the interaction which dominates the transition is $\sigma\phi^2$. The remaining terms

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correspond to quadratic or linear terms. They are present to *illustrate* an important feature of the relation of ϕ^4-theory to lower-dimensional models. The additional coupling λ in fact corresponds to the coupling constant of the two-dimensional $O(N)$ nonlinear σ model whose Lagrangian can be written in a form analogous to Eq. (1)

$$L = \frac{1}{2}\partial_\mu\phi_0^a\partial^\mu\phi_0^a + \frac{\sigma_0}{2}\left(\phi_0^a\phi_0^a - \frac{1}{\lambda_0}\right) , \tag{4}$$

which is renormalizable in two dimensions, but whose renormalization group functions can also be used to determine the critical exponents of the three-dimensional transitions. The point is that, at criticality, Eq. (3) is the full underlying theory. It is related to the boundary dimension models, where one reduces the space-time dimensionality from four, σ^2 becomes irrelevant in two dimensions whereas σ becomes relevant. On the contrary, when approaching four dimensions, σ^2 is relevant but σ becomes an irrelevant operator. In other words both models are equivalent at the appropriate critical point of their β-function. Thus either model can be used to determine critical exponents.

The power of the scaling dimension ξ in each term of Eq. (3) represents the anomalous dimension of that operator which is generated by radiative corrections in the quantum theory[a]. For instance, η is the ϕ-field anomalous dimension and is measured experimentally. In field theory, its numerical value is a reflection of the size of all radiative corrections. Therefore, by simple dimensional analysis, where α is the full dimension of ϕ, the results of Refs. [10,11] are

$$\alpha = \mu - 1 + \frac{1}{2}\eta, \tag{5}$$

where $d = 2\mu$. Likewise the anomalous dimension of the trivalent interaction is defined to be χ, giving [10,11]

$$\beta = 2 - \eta - \chi, \tag{6}$$

where β is the full σ field dimension. For the remaining two terms one finds the respective scaling laws

$$\chi_{\sigma^2} = 2\mu - 2\beta - 2\omega, \quad \beta = 2\mu - \frac{1}{\nu} . \tag{7}$$

[a]The term linear in σ does not have an associated anomalous operator dimension as this is already incorporated in its dimension β. The scaling law which arises from this term is recorded later.

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Having discussed the relation of the underlying theories and simply comparing Eqs. (2) and (3), there appears to be a connection with the coefficients of the kinetic operators. This is indeed the case which is readily established through the critical renormalization group equation and is documented, for example, in Ref. [2]. If the anomalous dimension is $\gamma(g)$, it is defined from the wave function renormalization constant through

$$\gamma(g) \;=\; \beta(g)\frac{\partial \ln Z_\phi}{\partial g}, \tag{8}$$

where $\beta(g) = \tilde{\mu}\partial g/\partial\tilde{\mu}$ is the usual β-function. Then the critical renormalization group equation gives in our conventions

$$\eta \;=\; \gamma(g_c), \tag{9}$$

where g_c is the d-dimensional non-trivial fixed point of the ϕ^4 β-function which represents the underlying phase transition of the model. The other exponents have analogous relations. For instance, in our conventions

$$\omega \;=\; -\;\frac{1}{2}\beta'(g_c), \quad \frac{1}{\nu} \;=\; -\;\beta'(\lambda_c), \tag{10}$$

where λ_c is the d-dimensional non-trivial fixed point of the $O(N)$ nonlinear σ model.

Therefore, having argued this relation to be between the usual renormalization constants and the critical exponents of the phase transition, one can provide the renormalization group functions to very high precision. The best current state is the five-loop work of Prof. Dr. Kleinert and collaborators [1]. Earlier calculations at lower orders were carried out in Refs. [12–16]. However, some initial attempts [15,16] at the five-loop calculation contained errors in several simple integrals which were observed and corrected in Ref. [1]. As a testimony to the huge calculation of Ref. [1], it is worth quoting the full five-loop $\overline{\mathrm{MS}}$ result for the d-dimensional β-function which is

$$\begin{aligned}
\beta(g) = (d-4)\frac{g}{2} \;&+\; [N+8]\frac{g^2}{6} \;-\; [3N+14]\frac{g^3}{6} \\
&+\; \left[33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)\right]\frac{g^4}{432} \\
&+\; \left[5N^3 - 6320N^2 - 80456N - 196648\right. \\
&-\; 96(63N^2 + 764N + 2332)\zeta(3) + 288(5N+22)(N+8)\zeta(4) \\
&-\; \left.1920(2N^2 + 55N + 186)\zeta(5)\right]\frac{g^5}{7776}
\end{aligned}$$

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$$
\begin{aligned}
+ \; & \big[13N^4 + 12578N^3 + 808496N^2 + 6646336N + 13177344 \\
& - \; 16(9N^4 - 1248N^3 - 67640N^2 - 552280N - 1314336)\zeta(3) \\
& - \; 768(6N^3 + 59N^2 - 446N - 3264)\zeta^2(3) \\
& - \; 288(63N^3 + 1388N^2 + 9532N + 21120)\zeta(4) \\
& + \; 256(305N^3 + 7466N^2 + 66986N + 165084)\zeta(5) \\
& - \; 9600(N+8)(2N^2 + 55N + 186)\zeta(6) \\
& + \; 112896(14N^2 + 189N + 526)\zeta(7)\big] \, \frac{g^6}{124416} \; + \; O(g^7), \qquad (11)
\end{aligned}
$$

where $\zeta(z)$ is the Riemann zeta function. The term $(d-4)g$, which corresponds to the dimension of the coupling, has been included to demonstrate the existence in d dimensions of a non-trivial value for g_c. This was used, together with the other renormalization group functions, to determine series for η, ν and ω, whose resummed three-dimensional values are in agreement with experiment and other methods [1].

3 Six Loops and Beyond

Given the need for the more accurate evaluation of critical exponents because of better experimental precision it is worth reviewing insights into the problem of tackling the extension of Eq. (11) to six loops and beyond. Two major approaches have recently been developed which attack different parts of the quintic polynomial in N which will appear as the six-loop coefficient of Eq. (11). The first of these is based on the observation that there appears to be a connection between abstract knot theory and number theory with the value of the Feynman diagrams when calculated in dimensional regularization. The initial breakthrough was by Kreimer in Ref. [17], where it was shown that the momentum routing in a Feynman graph could be associated with a knot link diagram. Skeining such link diagrams appropriately allowed one to decompose these into either a set of unknots or a set of unknots plus a prime knot. The remarkable and elegant feature which emerged was that the simple pole in ϵ of the corresponding Feynman diagram had only rational numbers in the former case but in the latter situation when a prime knot was involved, the Feynman diagram contained in addition to rationals a transcendental number such as $\zeta(3)$. In essence [17], if a Feynman graph skeined to a (prime) $(2,n)$-torus knot then its associated pole part contained $\zeta(n)$. This beautiful connection has since been studied extensively and the higher torus

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knots contain new zeta irreducible double and triple sums [18]. For instance, the next prime torus knot beyond the $(2,n)$-set is the $(3,4)$-torus knot which is associated in Feynman diagrams to the double sum

$$U_{6,2} = \sum_{n>m>0}^{\infty} \frac{(-1)^{n-m}}{n^6 m^2} . \tag{12}$$

This number had previously been investigated in Refs. [19,20] where it remained a puzzle since it could not be reduced to a series of products of ordinary $\zeta(n)$'s of the same level of transcendentality. In the knot context it turns out that the braid word structure of the associated prime knot has a simple correspondence with the nested sum structure which is not reducible to lower $\zeta(n)$'s [17,18]. Whilst these revolutionary ideas were developed without reference to a particular field theory or its symmetry properties, it was not clear whether such new zeta irreducible numbers would in fact arise in the renormalization group functions of a four-dimensional theory. However, it was shown in ϕ^4-theory for $N=1$ in Ref. [18] that certain diagrams at six (and seven) loops with no subgraph divergences had a non-trivial knot number structure beyond the $(2,n)$-torus knot level. Indeed $U_{6,2}$ arose in that part of the six-loop β-function polynomial which was N-independent and its coefficient was calculated explicitly. Therefore, computing higher-order corrections to Eq. (11) would have to account for this new feature. For instance, knowing that such structures will exist could allow one to exploit it as a basis for performing such calculations.

The second method of gaining insight into the form of the six-loop and higher $\overline{\text{MS}}$ β-function is to determine the coefficients of the leading and next to leading terms of the polynomial in N at each loop order. This is provided by the large N method developed originally for the $O(N)$ σ model in Refs. [10,11,21]. There the d-dimensional critical exponents themselves were computed in successive powers of $1/N$ to three terms in the series for η and ν. Since they are expressed as functions of $d=2\mu$, one can extract through the critical renormalization group equation information on the coefficients of the corresponding renormalization group functions in $4-2\epsilon$ dimensions and compare it with the explicit $\overline{\text{MS}}$ perturbative results as a consequence of the critical point equivalence. To the orders each of these is computed to, there is exact agreement between both. More importantly, this connection can be used to gain information on the coefficients of the renormalization group functions going beyond those currently calculated at the orders in $1/N$ which are

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available. To achieve this for four-dimensional ϕ^4-theory, one requires knowledge of the location of the fixed point g_c in d dimensions at the appropriate order in $1/N$. This therefore requires the critical exponent ω at $O(1/N^2)$ which was calculated in Ref. [22]. The method is based on the Lagrangian of the form

$$L \;=\; \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a \;+\; \frac{1}{2}\sigma\phi^a\phi^a \;-\; \frac{3\sigma^2}{32\pi^2 g}. \tag{13}$$

It is used to apply the uniqueness method of integration [10,11] when computing the large set of Feynman diagrams which occur at $O(1/N^2)$. Thus, one can determine the critical exponent ω as [22]

$$\omega_1 \;=\; (2\mu-1)^2\eta_1 \tag{14}$$

and

$$\begin{aligned}
\omega_2 = -\; & \left[\frac{4(\mu^2-5\mu+5)(2\mu-3)^2(\mu-1)\mu^2[\bar\Phi(\mu)+\bar\Psi^2(\mu)]}{(\mu-2)^3(\mu-3)} \right. \\
+\; & \frac{16\mu(2\mu-3)^2}{(\mu-2)^3(\mu-3)^2\eta_1} \\
+\; & \frac{3(4\mu^5-48\mu^4+241\mu^3-549\mu^2+566\mu-216)(\mu-1)\mu^2\hat\Theta(\mu)}{2(\mu-2)^3(\mu-3)} \\
+\; & [16\mu^{10}-240\mu^9+1608\mu^8-6316\mu^7+15861\mu^6 \\
& -25804\mu^5+26111\mu^4-14508\mu^3+2756\mu^2 \\
& +672\mu-144)]/[(\mu-2)^4(\mu-3)^2]\bar\Psi(\mu) \\
-\; & [144\mu^{14}-2816\mu^{13}+24792\mu^{12}-130032\mu^{11}+452961\mu^{10} \\
& -1105060\mu^9+1936168\mu^8-2447910\mu^7+2194071\mu^6 \\
& -1320318\mu^5+460364\mu^4-43444\mu^3-26280\mu^2 \\
& \left. +8208\mu-864]/[2(2\mu-3)(\mu-1)(\mu-2)^5(\mu-3)^2\mu] \right] \eta_1^2,
\end{aligned} \tag{15}$$

where $\omega = \sum_{i=0}^\infty \omega_i/N^i$ and $\omega_0 = \mu - 2$. The various variables and functions are defined by

$$\begin{aligned}
\eta_1 &= \frac{2(\mu-2)\Gamma(2\mu-1)}{\mu\Gamma(1-\mu)\Gamma^3(\mu)}, \\
\bar\Psi(\mu) &= \psi(2\mu-3) \;+\; \psi(3-\mu) \;-\; \psi(\mu-1) \;-\; \psi(1), \\
\hat\Theta(\mu) &= \psi'(\mu-1) \;-\; \psi'(1),
\end{aligned}$$

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$$\bar{\Phi}(\mu) = \psi'(2\mu-3) \;-\; \psi'(3-\mu) \;-\; \psi'(\mu-1) \;+\; \psi'(1), \tag{16}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ and $\Gamma(x)$ is the Euler Γ-function. So, for example, if we represent all orders $\overline{\mathrm{MS}}$ ϕ^4 β-function at $O(1/N^2)$ as

$$\beta(g) \;=\; \tfrac{1}{2}(d-4)g \;+\; (a_1N+b_1)g^2 \;+\; \sum_{r=2}^{\infty}(a_rN+b_r)N^{r-2}g^{r+1}, \tag{17}$$

then we find the *new* $\overline{\mathrm{MS}}$ coefficients

$$\begin{aligned}
a_6 &= [29 \;+\; 528\zeta(3) \;-\; 432\zeta(4)]/1866240,\\
a_7 &= [61 \;+\; 80\zeta(3) \;+\; 1584\zeta(4) \;-\; 1728\zeta(5)]/26873856,\\
a_8 &= -\;[5760\zeta(6) \;-\; 6336\zeta(5) \;-\; 240\zeta(4)\\
&\qquad +\; 1152\zeta^2(3) \;-\; 208\zeta(3) \;-\; 125]/376233984,\\
b_6 &= -\;[28160\zeta(7) \;-\; 95200\zeta(6) \;+\; 150336\zeta(5) \;+\; 6912\zeta(4)\zeta(3)\\
&\qquad -\; 14112\zeta(4) \;-\; 24064\zeta^2(3) \;-\; 11880\zeta(3) \;-\; 5661]/466560,\\
b_7 &= -\;[8520960\zeta(8) \;-\; 32724480\zeta(7) \;+\; 43286400\zeta(6) \;+\; 3993600\zeta(5)\zeta(3)\\
&\qquad -\; 31998720\zeta(5) \;-\; 8663040\zeta(4)\zeta(3) \;-\; 2538432\zeta(4)\\
&\qquad +\; 11381760\zeta^2(3) \;+\; 7461168\zeta(3) \;+\; 1125439]/403107840,\\
b_8 &= -\;[9210880\zeta(9) \;-\; 41166720\zeta(8) \;+\; 61054080\zeta(7) \;+\; 4300800\zeta(6)\zeta(3)\\
&\qquad -\; 38500800\zeta(6) \;+\; 2995200\zeta(5)\zeta(4) \;-\; 19553280\zeta(5)\zeta(3)\\
&\qquad +\; 11519040\zeta(5) \;+\; 17072640\zeta(4)\zeta(3) \;+\; 5863104\zeta(4)\\
&\qquad +\; 542720\zeta^3(3) \;-\; 10141440\zeta^2(3)\\
&\qquad -\; 4518336\zeta(3) \;-\; 717083]/1410877440\,.
\end{aligned} \tag{18}$$

Similarly, if one represents the $\overline{\mathrm{MS}}$ field anomalous dimension at $O(1/N^3)$ as

$$\gamma(g) \;=\; \sum_{r=1}^{\infty}(c_rN^2+d_rN+e_r)N^{r-2}g^{r+1}, \tag{19}$$

where $e_1 \equiv 0$, the large N results give

$$\begin{aligned}
e_9 = [&1560674304\zeta(10) \;-\; 12534896640\zeta(9) \;+\; 11070010560\zeta(8)\\
&+ 1732018176\zeta(7)\zeta(3) \;+\; 581961984\zeta(7) \;-\; 3411394560\zeta(6)\zeta(3)\\
&- 2684240640\zeta(6) \;+\; 209534976\zeta^2(5) \;-\; 1567752192\zeta(5)\zeta(4)\\
&+ 1754664960\zeta(5)\zeta(3) \;-\; 975533568\zeta(5) \;-\; 9289728\zeta(4)\zeta^2(3)\\
&+ 1310201856\zeta(4)\zeta(3) \;+\; 1636615872\zeta(4) \;-\; 137158656\zeta^3(3)
\end{aligned}$$

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$$- 1708996608\zeta^2(3) + 294403968\zeta(3)$$
$$- 89800704U_{62} - 341350433]/1950396973056 . \qquad (20)$$

The previous term in the series at this order, e_8, was given in Ref. [23] and like the expression for e_9, it contains the zeta irreducible $U_{6,2}$. However, it is important to recognise that the first appearance of this number in the *full* anomalous dimension will be at a lower-loop order than the ninth order of e_8.

4 Discussion

Whilst the knot theory insight into the higher-order structure of the $\overline{\mathrm{MS}}$ ϕ^4 four-dimensional renormalization group functions is quite impressive, true progress in this area will only be represented by the provision of the *full* result at six loops. This would require a huge amount of tedious computation since, for instance, one needs to determine the *finite* part of the large number of five-loop diagrams as they will contribute when multiplied by the one-loop vertex counterterms. Therefore, we believe such a result will not appear in the foreseeable future and hence Eq. (11) remains the current state of the art. Nevertheless, to emphasise Prof. Dr. Kleinert's continued interest and impressive contribution to this field, it is worth mentioning an extension of the calculation of Eq. (11) to a model which involves, in addition to an $O(N)$ ϕ^4-interaction, a cubic interaction. The critical exponents for this double coupling model were computed again to five loops in $\overline{\mathrm{MS}}$ in Ref. [24] to explore in detail the stability of a variety of fixed points which occur in this model since they correspond to phase transitions in three-dimensional cubic crystals.

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