Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

WHAT CAN ISING SPINS TEACH US ABOUT QUANTUM GRAVITY?

C. HOLM

Max-Planck-Institut für Polymerforschung,
Ackermannweg 10, D-55128 Mainz, Germany
E-mail: holm@mpip-mainz.mpg.de

We investigate numerically an Ising spin model coupled to two-dimensional Euclidean quantum gravity. We employ Regge calculus to discretize the gravitational interaction. We study this system on a toroidal and a spherical manifold, with two different local path integral measures, and an added R^2 interaction term. We find in all cases that the critical exponents of the Ising transition are consistent with the Onsager values, and that the KPZ exponents are definitely excluded.

1 Introduction

The study of two-dimensional (2D) models has often proved to be an important first step in developing fundamental ideas concerning higher-dimensional physics. Such has happened for 2D Euclidean quantum gravity where we have by now independent analytic results from conformal field theory [1] and matrix models [2], for the critical exponents of a simple toy matter field, namely a spin model coupled to a fluctuating geometry. Historically, the interest in 2D quantum gravity was inspired by string theory. The time development of strings leads to 2D world surfaces, which are reparametrization invariant and quantized, hence describe 2D quantum gravity. Kazakov [3] suggested a model of Ising spins living on the vertices of so-called planar ϕ^4 graphs. An exact expression of the partition function was given for Z_n in the thermodynamic limit $n \to \infty$ by relating this model to an exactly solvable Hamiltonian model of two Hermitian matrices [2,4]. Shortly afterwards [5], the same model was solved also on ϕ^3 graphs, and some time later Knizhnik, Polyakov, and

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Zamolodchikov (KPZ) [1] found the same set of critical exponents in a continuum model of 2D quantum gravity using methods of conformal field theory for matter of central charge $c = 1/2$. Since continuum field theory as well as lattice models agree on the set of critical exponents, one felt comfortable with the idea that Kazakov's model was really a model of quantum gravity, and that matter fields can be strongly influenced, when they are coupled to a quantum geometry. These new critical exponents were called KPZ exponents, and turn out to be quite different from the original Ising exponents. The critical exponents change from the flat-space Onsager values $\alpha = 0$, $\beta = 0.125$, $\gamma = 1.75$, and $\nu = 1$ to the values $\alpha = -1$, $\beta = 0.5$, $\gamma = 2$, and $D\nu = 3$, where D is the internal fractal dimension of the manifold.

In the following work we will entirely remain in two dimensions and investigate if the analytic KPZ results obtained for the Ising system ($c = 1/2$) can be obtained with Regge's method transcribed to the quantum domain. Although this question poses itself rather naturally, only few people have actually investigated this subject. The first to look at this problem were Gross and Hamber [6] who found the classical flat-space Ising critical exponents. This came as a surprise since a different method, that was termed dynamical triangulated random surfaces (DTRS) and that is more or less a Monte Carlo version of the Boulatov model, gave KPZ results. We have put considerably effort in modifying the global topology, the local path integral measure, and added an R^2 interaction term, in order to see if one observes any effect on the critical exponents. The results we will present here, have been obtained over the course of the past seven years [7,8]. We will first review the method, then the simulation technique, followed by the results, and end with some conclusion.

2 Regge Calculus

Regge calculus [9] is a discretization approach to gravity which reduces the infinite degrees of freedom of Riemannian manifolds to a finite number of parameters by working with piecewise linear spaces. Regge calculus has found numerous applications in classical and quantum physics. An introduction with an extensive list of references can be found in Ref. [10]. It can undoubtedly be regarded as the best-understood method to discretize classical gravity. Historically this method was used in the first numerical attempts to study quantum gravity non-perturbatively [11,12], but it can also be used as a regularized version of quantum gravity in which one can perform analytic

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

calculations [13]. It has been mostly employed in four dimensions where extensive simulations have been carried out [14]. The Regge approach in two dimensions consists of choosing a triangulation of the manifold under consideration, which means that the topology stays fixed from the beginning. One then assigns link lengths to each triangle (or simplex, in higher dimensions), which play the role of the dynamical variables. This is, incidentally, the opposite procedure to what one does in the so-called DTRS method, where one has fixed edge lengths, and a fluctuating connectivity. The values of its squared edge length $q_{ij} = l_{ij}^2$ induce a constant metric in the interior of each simplex, because q_{ij} can linearly be related to the three components of the metric $\eta_{\mu\nu}$. Local curvature can be described by the rotation experienced by a vector when it is parallel transported in a closed curve around a vertex, where several triangles meet. The angle of rotation is measured by the deficit angle δ_i at the vertex i that can intrinsically be defined as

$$\delta_i = 2\pi - \sum_{\text{all } t \text{ sharing } i} \theta_i(t), \tag{1}$$

and $\theta_i(t)$ is the dihedral angle associated with the triangle t. Curvature is therefore distributed delta-function-like with support on the vertices. The dihedral angles can be computed purely out of the link length information. Defining the barycentric area connected to the site i,

$$A_i = \sum_{t \supset i} \frac{1}{3} A_t, \tag{2}$$

where A_t denotes the area of the triangle t, one can then write down the simplicial analogue of continuum integrals like

$$\int d^2x \sqrt{g(x)} \longrightarrow \sum_i A_i, \tag{3}$$

$$\frac{1}{2} \int d^2x \sqrt{g(x)} R(x) \longrightarrow \sum_i \delta_i = 2\pi\chi(\mathcal{M}), \tag{4}$$

$$\int d^2x \sqrt{g(x)} R^2(x) \longrightarrow 4 \sum_i \frac{\delta_i^2}{A_i}. \tag{5}$$

Eq. (4) is the simplicial analogue of the Gauss-Bonnet theorem which relates the differential-geometric integral on the left hand side to a topological invariant, namely the Euler characteristic χ. This makes pure 2D gravity, based on the action corresponding to Eq. (4), dynamically trivial, but leaves still

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

open the possibility that matter fields can be influenced by the fluctuating geometry. In two dimensions it can be shown that Eq. (5) gives the exact continuum result on any regular triangulation of the sphere [11].

The Euler characteristic of a two-dimensional manifold $\mathcal{M}$ can also be written as $\chi = 2(1-g)$, where g is the gender of the surface, which counts how many holes there are in $\mathcal{M}$. For a simplicial complex, the Euler characteristic can also be computed as

$$N_0 - N_1 + N_2 = 2(1-g), \tag{6}$$

where N_0, N_1, and N_2 denote the number of sites, links, and triangles, respectively. For a compact complex we also know that a link is shared by two triangles, resulting in the relation $N_1/3 = N_2/2$. From these two relations one can derive two more, namely $N_0 - 2(1-g) = N_2/2$ and $N_0 - 2(1-g) = N_1/3$, which will become useful later. The sphere has $g = 0$ and the torus has $g = 1$.

3 Model and Simulation Techniques

We simulated the partition function

$$Z = \sum_{\{s\}} \int D\mu(l) \exp\left[-I(l) - KE(l,s)\right] \delta\left(\sum_i A_i - A\right), \tag{7}$$

where $\{s\}$ denotes the set of all spin configurations of the Ising spins $s_i = \pm 1$, and the gravitational action is defined as

$$I(l) = \sum_i \left(\lambda A_i + a\frac{\delta_i^2}{A_i}\right). \tag{8}$$

The energy of Ising spins, which are located at the vertices i of the lattice, is denoted by

$$E(l,s) = \frac{1}{2} \sum_{\text{edges } l_{ij}} A_{ij} \left(\frac{s_i - s_j}{l_{ij}}\right)^2, \tag{9}$$

and the barycentric area A_{ij} associated with a link l_{ij} is defined as

$$A_{ij} = \sum_{\text{triangles } t \supset l_{ij}} \frac{1}{3} A_t. \tag{10}$$

The energy is the discretized analogue of the continuum action for a scalar field φ, i.e. $\int d^2x \sqrt{g(x)} g^{\mu\nu}(x) \partial_\mu \varphi \, \partial_\nu \varphi$.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

The delta function in Eq. (7) ensures that the total area is kept fixed at a given value A, thus rendering the cosmological constant irrelevant, and classically, gravitational dynamics can only arise from an R^2-interaction term. Such a term was used in two and higher dimensions to cure the unboundedness problem of the pure gravitational action [11]. From dimensional arguments, one would expect such a term to be irrelevant for the Ising transition. An inclusion of this term enabled us to test these arguments.

The quantization procedure of the classical Regge action rests on the path integral formulation [15] of Eq. (7). For the path-integral measure $D\mu(l)$ we mostly used a simple scale-invariant measure of the form [7,16]

$$\mathcal{D}\mu(q) = \prod_{\langle ij \rangle} dq_{ij}/q_{ij} F_\epsilon(\{q_{ij}\}). \tag{11}$$

The function $F_\epsilon(\{q_{ij}\})$ ensures Euclidean geometry. If the triangle inequalities are obeyed it assumes the value one, otherwise it vanishes. The parameter ϵ serves to suppress very thin triangles by generalizing the triangle inequalities to a (still scale-invariant) form $(l_1 + l_2) \geq (1-\epsilon) l_3$ and $(l_1 - l_2) \leq (1+\epsilon) l_3$. This makes the algorithm faster, because many proposed new values for l that would get discarded with high probability, are thrown out at some earlier program steps. This is not necessary for convergence, unlike in higher dimensions. For our simulations, ϵ was of the order 10^{-4}. We checked that a different value of ϵ did not change the outcome of our measurements. The attractiveness of the measure (11) lies in the fact that it is local, scale invariant, and the integral runs directly over the variables $\xi_{ij} = \ln l_{ij}$, which makes it well-adapted for computer simulations, hence the term "computer measure". Sometimes we will also use the abbreviation "dl/l-measure" for the measure of Eq. (11).

Most continuum measures differ by the power of the determinant of the metric g, which stands in front of the integration measure. In a naive transcription of the volume element $\sqrt{g}$ to the Regge formalism one identifies $\sqrt{g} \longrightarrow A_{ij}$, where A_{ij} is a volume associated with a link l_{ij}. Keeping the freedom of having some power of l_{ij} appearing in the measure, one is led to consider the following two-parameter class of measures [6]:

$$\mathcal{D}\mu(q) = \prod_{\langle ij \rangle} A_{ij}^{2\sigma} q_{ij}^{(\alpha-1)} dq_{ij} F_\epsilon(\{q_{ij}\}). \tag{12}$$

In terms of the two parameters α and σ the computer measure corresponds to $\alpha = 0$, $\sigma = 0$. An analog of the scale invariant Misner measure would be

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

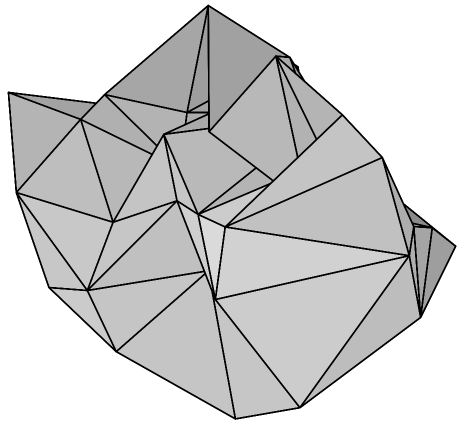

Figure 1. dl/l measure configuration for the sphere with $d = 4$, corresponding to $N_0 = 56$.

$\alpha = 1, \sigma = -1/2$, and the lattice analog of the popular DeWitt measure is given by $\alpha = 1, \sigma = -1/4$. We will present here also some preliminary data with the DeWitt measure, which is not scale-invariant [8].

We have simulated the gravitational action using a standard multi-hit Metropolis update. As Ising update we used the single-cluster (Wolff) algorithm [17] which prevents the critical slowing down near the phase transition. One update in the single-cluster variant consists of choosing a random mirror plane and a random site, which is the starting point for growing a cluster of reflected spins. The size and shape of the cluster is controlled by a Metropolis like accept/reject criterion satisfying detailed balance.

Between measurements we performed $n = 2, \ldots, 4$ Monte Carlo steps consisting of one lattice sweep to update the link lengths l_{ij} followed by a single-cluster flip to update (a fraction of) the spins s_i. We tested in some cases that varying the relative frequency of link and spin updates does not change the results within error bars.

We used two different global topologies for our simulations to check for a possible influence of topology on the phase transition. The main investigation was performed on regular triangulated tori of size $N_0 = L^2$ with fixed coordination number $q = 6$. This triangulation gives rise to $2N_0$ triangles and $3N_0$ link variables. The principal simulations were performed at a curvature squared coupling value of $a = 0.001$ and the couplings $K = 1$ and $K = 1.025$ for $L = 6, 8, 10, 12, 16, 32, 64, 100, 128,$ 200, 256, and 512. Additional simulations were performed with $a = 0$ and 0.1 at $K = 1.025$, using lattices of

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

size $L = 8, 16, 32, 64, 100, 128$, and 256. Because of the scale invariance of the measure we could rescale each link when proposing a link update such that the total area was kept fixed to its initial value $A = N_0$. The difference of the model defined by (7) and the Ising model on a static triangular lattice is that the spins are coupled by geometric weight factors $w_{ij} = A_{ij}/l_{ij}^2$ which can fluctuate around the static value $w_{ij} = \sqrt{3}/6$.

For the spherical lattice topology we used the triangulated surface of a three-dimensional cube of edge length d. This provides us with an almost regular triangulation of the sphere where six vertices have coordination number four, and all others have coordination number six. In terms of the linear length d of the cube the number of vertices is $N_0 = 6(d-1)^2 + 2$. For further reference the number of links and triangles in terms of N_0 are given by $N_1 = 3N_0 - 6$, and $N_2 = 2N_0 - 4$, respectively. We studied ten system sizes ranging from $d = 10\,(N_0 = 488)$ up to $d = 55\,(N_0 = 17498)$. The area was kept fixed to its initial value $A = N_2/2$, and we used no coupling to the R^2 term, i.e. $a = 0$. As simulation point we have chosen a value of $K_0 = 1.025$, already anticipating that this value is close to the critical coupling K_c on the torus. To compare previous values we set as our linear length scale $L = \sqrt{N_0}$. A typical configuration can be viewed at in Fig. 1, which was produced using the computer measure of Eq. (11)

For each run we recorded the time series of the energy density $e = E/N_0$, the magnetization density $m = \sum_i A_i s_i/N_0$ and the Liouville field density $\varphi = \sum_i \ln A_i/N_0$. After an initial equilibration time, we performed for each lattice size about 50 000 measurements. From an analysis of the time series we found integrated autocorrelation times for the energy and the magnetization of about $1-7$ (in units of measurements) for all lattice sizes. To obtain results for the various observables $\mathcal{O}$ at K values in an interval around the simulation point K_0, we applied the reweighting method [18]. Since we recorded the time series this amounts to computing

$$\langle \mathcal{O} \rangle|_K = \frac{\langle \mathcal{O} e^{-\Delta K E} \rangle|_{K_0}}{\langle e^{-\Delta K E} \rangle|_{K_0}}, \tag{13}$$

with $\Delta K = K - K_0$. To obtain errors we divided each run into 20 blocks and computed standard Jackknife errors. At $a = 0.001$ where we had two simulations at different K values, we combined the results according to their errors [19].

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

From the time series we computed the Binder parameter [20],

$$U_L(K) = 1 - \frac{1}{3}\frac{\langle m^4\rangle}{\langle m^2\rangle^2}. \tag{14}$$

It is well known that the $U_L(K)$ curves for different L cross around (K_c, U^*) with slopes $\propto L^{1/\nu}$, apart from confluent corrections explaining small systematic deviations. This allows an almost unbiased estimate of the critical coupling K_c, the critical correlation length exponent ν, and the renormalized charge U^*. We further analyzed the (finite lattice) susceptibility,

$$\chi(K) = N_0(\langle m^2\rangle - \langle |m|\rangle^2), \tag{15}$$

the susceptibility in the disordered phase,

$$\chi'(K) = N_0(\langle m^2\rangle), \tag{16}$$

the specific heat,

$$C(K) = K^2 N_0(\langle e^2\rangle - \langle e\rangle^2), \tag{17}$$

and studied the (finite lattice) magnetization at its point of inflection, $\langle |m|\rangle|_{\mathrm{inf}}$. The inflection point can be obtained from the maximum of $d\langle |m|\rangle/dK$. Further useful quantities are the logarithmic derivatives $d\ln\langle |m|\rangle/dK$ and $d\ln\langle m^2\rangle/dK$. Another gravitational quantity of interest is the Liouville field $\varphi(x) = \ln\sqrt{g(x)}$. In the discretized version its lattice average reads as $\varphi = 1/N_0\sum_i \ln A_i$, and the associated lattice Liouville susceptibility is defined as $\chi_\varphi(L) = N_0(\langle \varphi^2\rangle - \langle\varphi\rangle^2)$.

4 Results

By applying the outlined reweighting techniques we first determined the maxima of χ, C, $d\langle |m|\rangle/dK$, $d\ln\langle |m|\rangle/dK$, and $d\ln\langle m^2\rangle/dK$. The location of the maxima provided us with five sequences of pseudo-transition points $K_{\max}(L)$ for which the scaling variable $x = (K_{\max}(L) - K_c)L^{1/\nu}$ should be constant. Using this information we then have several possibilities to extract the critical exponent ν from (linear) least square fits of the finite-size scaling (FSS) ansatz $dU_L/dK \cong L^{1/\nu} f_0(x)$ or $d\ln\langle |m|^p\rangle/dK \cong L^{1/\nu} f_p(x)$ to the data at the various $K_{\max}(L)$.

For the torus, the very extensive simulations at $a = 0.001$ show good agreement with the Onsager value $\nu = 1$ at a 2 % level. For the other

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

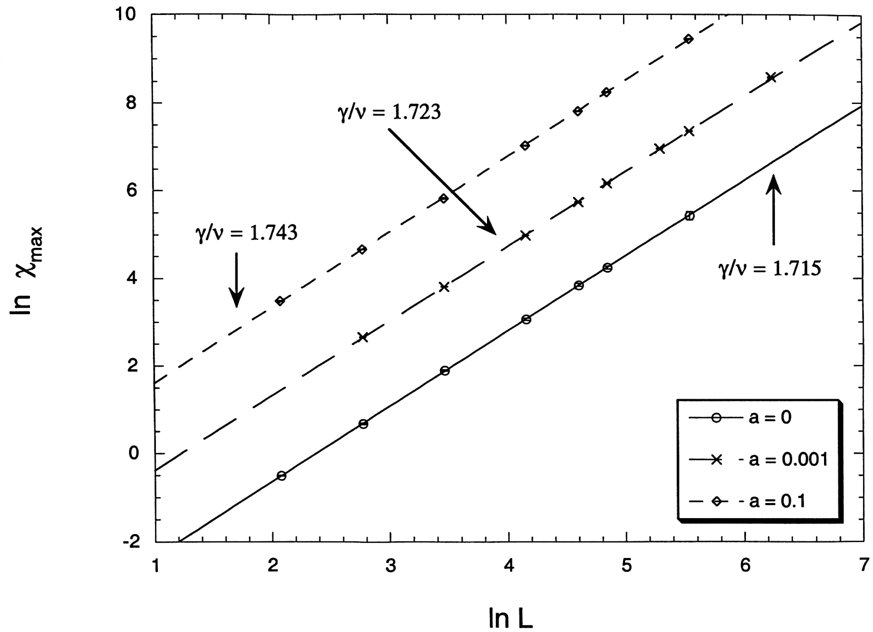

Figure 2. Double logarithmic finite-size scaling plot of the susceptibility maxima $\chi_{\max}$ for $a = 0.0, 0.001$, and 0.1 on the torus. To disentangle the curves we added an offset of -2 (2) to the data for $a = 0.0$ ($a = 0.1$). The slopes are in all three cases compatible with the Onsager value $\gamma/\nu = 1.75$ for regular static lattices.

two couplings, $a = 0.1$ and $a = 0$, the data scatter a bit more but are still compatible with $\nu = 1$. For the sphere the values are slightly smaller, but still compatible with one. Combining this information we use in further analyses

$$K_c = 1.0234 \pm 0.0002 \quad (a = 0.0), \text{ Torus,} \tag{18}$$
$$K_c = 1.0230 \pm 0.0010 \quad (a = 0.0), \text{ Sphere,} \tag{19}$$
$$K_c = 1.0265 \pm 0.0001 \quad (a = 0.001), \text{ Torus,} \tag{20}$$
$$K_c = 1.0295 \pm 0.0001 \quad (a = 0.1), \text{ Torus.} \tag{21}$$

In particular we can now test the consistency of our data and extract ν also from the scaling of dU/dK and $d\ln\langle|m|^p\rangle/dK$ at K_c, see Table 1.

To extract the critical exponent ratio γ/ν we used the scaling $\chi \cong L^{\gamma/\nu} f_3(x)$ at the previously discussed points of constant x, as well as the

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Table 1. Comparison of our Monte Carlo results on the torus and on the sphere with the exact results for the Ising model on static lattices (Onsager) and the KPZ exponents. The values marked with a star were computed from hyperscaling relations with $D = 2$, thereby neglecting possible scaling effects due to the internal fractal dimension in the DTRS approach.

	α	β	γ	δ	η	ν
KPZ	-1	0.5	2	5	$2/3^*$	1.5^*
Onsager	0	0.125	1.75	15	0.25	1
Torus-dl/l	≈ 0	0.126(2)	1.75(2)	14.9(3)	0.272(3)	1.01(1)
Sphere-dl/l	≈ 0	0.130(26)	1.61(12)	14(3)	0.256(6)	0.93(5)
Sphere-DeWitt	≈ 0	0.12(1)	1.75(1)	15(2)	0.25(1)	1.00(1)

scaling of χ' at K_c. The values for γ/ν for the different values of a are compatible with each other, but are all slightly below the Onsager value of $\gamma/\nu = 1.75$. Due to their respective error range, however, they are still consistent with the flat space exponent ratio. The quality of the fits for $\chi_{\max}$ on the torus can be inspected in Fig. 2, and the final values for γ, inserting our previously determined value for ν, can be found also in Table 1. For the sphere a weighted fit over all values gave $\gamma/\nu = 1.744(6)$.

To extract the magnetical critical exponent ratio β/ν we used that $\langle|m|\rangle \cong L^{-\beta/\nu} f_4(x)$ at all constant x-values. Another method is to look at the scaling of $d\langle|m|\rangle/dK \cong L^{(1-\beta)/\nu} f_5(x)$. Because the errors on the different estimates turned out to vary over a large range we chose to compute error-weighted averages. Using our average values for ν in Table 1 we obtain the final estimates of $\beta/\nu = 0.127(3)$ (torus, $a = 0.001$), $\beta/\nu = 0.123(2)$ (torus, $a = 0.1$), $\beta/\nu = 0.123(4)$ (torus, $a = 0.0$), and $\beta/\nu = 0.14(2)$ (sphere, $a = 0.0$). Again we see little influence of the curvature square term, and the results are again in agreement with the Onsager result $\beta/\nu = 0.125$.

For a specific-heat exponent α of zero we expect a logarithmic divergence like

$$C(x, L) = A(x) + B(x) \ln L. \tag{22}$$

Indeed the data at the different fixed values of x could all be fitted nicely with this ansatz, supporting again the Onsager value. We also performed simulations using a lattice transcription of the DeWitt measure according to Eq. (12) with $\alpha = 1, \sigma = -1/4$, which did not change the results at all [8], and some results from a preliminary data analysis can be found in Table 1 as well,

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

where we also used the scaling relations $\eta = 2 - \gamma/\nu$, and $\delta = 1 + \gamma/\beta$. The short summary of our findings is that the critical exponents still agree with the Onsager exponents for regular static lattices to a high degree of accuracy, and KPZ exponents are definitely excluded.

For the sphere, the non-regular triangulation seems to affect the finite-size behavior in a negative way, and one could probably obtain more accurate results comparable to those on the torus, by using a random triangulation of the sphere [21,22]. Unlike in the pure gravity case, where the global lattice topology enters in the formula for the string susceptibility exponent [23], it does not affect the critical exponents of the Ising phase transition.

5 Conclusions

Using a highly efficient cluster update algorithm and advanced reweighting techniques, our results have shown that the model defined by (7), describing Ising spins coupled to quantum gravity, remained in the Onsager universality class. This statement still holds if the global lattice topology was changed from the torus to the sphere. We conclude therefore that the global topology does not play any role for the Ising critical exponents, as it does for example for the string susceptibility exponent γ_{str}. We have also tested two local lattice measures, the dl/l and DeWitt measure, and saw no effect on the Ising transition. We have also found no influence of an added curvature square term R^2, as one would expect from dimensional arguments. Overall we can conclude that neither the global topology nor the change of the local measure or the added R^2 interaction term change the critical exponents of the Ising system coupled to gravity via Eq. (9). The KPZ exponents are definitely excluded. We have further shown that one can use the Ising system ($c = 1/2$) as a probe to test KPZ scaling because there the FSS analyses are standard and give very accurate results.

Unfortunately the present situation is unsatisfactory, because one needs to explain why different discrete approaches to a “simple” 2D quantum gravity model lead to different results. After all, both models were supposed to describe the same continuum physics. There are still various possibilities to explain this:

- one needs to use a discrete nonlocal measure,
- the spin coupling to gravity is not correct,
- KPZ results are due to the fluctuating incidence matrix of the lattice,

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

- KPZ results are due to Euclidean gravity.

The first point was advocated strongly by Menotti and others [24], however, up to today nobody has shown that with a non-local measure one is able to get the KPZ exponents using the Regge calculus formulation. There are also calculations which show [25] that the proposed non-local lattice measures fail to agree with their continuum counterparts already in the weak field, low momentum limit, hence are not acceptable discrete functional measures. The second point relies on the fact that the coupling of the Ising spins to gravity in the Regge formulation was only heuristically written down in Eq. (9). A more profound investigation might lead to a different lattice implementation, which in turn could change the critical behavior. However, nobody has yet investigated this possibility. The third point suggests that the Regge method, because it is based on a fixed incidence matrix, cannot capture the necessary randomness in the coordination number, to induce a change of universality class. This randomness might, however, be of such an importance only in two dimensions, because here the Einstein-Hilbert term is trivial, and all what is left is the freedom of the incidence matrix to rearrange. In higher dimensions one can assume that the propagation modes of the Einstein-Hilbert term will be dominant so that this possible drawback of Regge calculus becomes unimportant. It is nevertheless unclear, why a purely field theoretic continuum Lagrangian like the one in Ref. [1] should give different results from the Regge method which is designed to approximate just the same model.

The last point goes back to suggestions by Ambjørn *et al.* [26] who showed that even for the DTRS method the critical exponents actually remain in the Onsager universality class if one uses a Lorentzian gravity formulation instead of an Euclidean one. The Lorentzian space-time structure and its causal requirement seems to be so stringent that it smoothes the possible randomness, thereby leaving the Ising system in its flat-space class. Regge calculus was actually designed to work in a Lorentzian geometry, but this still does not explain why it apparently is not able to describe well Euclidean geometry. This might again be related to the lack of the necessary randomness due to the fixed incidence matrix.

We still think that the question why the Regge method shows no effect on the Ising transition has to be resolved completely, because in one way or the other, we will learn lessons for future studies of a more realistic theory of quantum gravity. Even though two-dimensional gravity is, classically spoken, rather trivial, these questions await still their final answer.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Acknowledgments

The work described here was performed in a pleasant and close collaboration with W. Janke. I would like to thank H. Kleinert for enjoyable discussions and for providing a stimulating environment during my time in Berlin. I sincerely wish him many more fruitful years to come.

References

[1] V.G. Knizhnik, A.M. Polyakov, and A.B. Zamolodchikov, *Mod. Phys. Lett.* **A3**, 819 (1988).

[2] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, *Phys. Rep.* **254**, 1 (1995); P. Ginsparg, in *Proceedings of the 1991 Summer School in High Energy Physics and Cosmology*, ICTP Series in Theoretical Physics, Vol. 8, Ed. E. Gava *et al.* (World Scientific, Singapore, 1992).

[3] V.A. Kazakov, *JETP Lett.* **44**, 133 (1986); *Phys. Lett. A* **119**, 140 (1986).

[4] C. Itzykson and J.B. Zuber, *J. Math. Phys.* **21**, 411 (1980); M.L. Metha, *Comm. Math. Phys.* **79**, 327 (1981).

[5] D.V. Boulatov and V.A. Kazakov, *Phys. Lett. B* **186**, 379 (1987).

[6] M. Gross and H.W. Hamber, *Nucl. Phys. B* **364**, 703 (1991).

[7] C. Holm and W. Janke, *Phys. Lett. B* **335**, 143 (1994); *Nucl. Phys. B* **42** (Proc. Suppl.), 725 (1995); *Phys. Lett. B* **375**, 69 (1996).

[8] C. Holm and W. Janke, preprint.

[9] T. Regge, *Nuovo Cim.* **19**, 558 (1961).

[10] R.M. Williams and P.A. Tuckey, *Class. Quantum Grav.* **9**, 1409 (1992). This bibliographic review contains an almost complete list of references about Regge calculus up to 1991.

[11] H.W. Hamber and R.M. Williams, *Nucl. Phys. B* **248**, 392 (1984); *Phys. Lett. B* **157**, 368 (1985); *Nucl. Phys. B* **267**, 482 (1986).

[12] B.A. Berg, *Phys. Rev. Lett.* **55**, 904 (1985); *Phys. Lett. B* **176**, 39 (1986).

[13] M. Rocek and R.M. Williams, *Phys. Lett. B* **104**, 31 (1981); *Z. Phys. C* **21**, 371 (1984).

[14] H.W. Hamber, *Nucl. Phys. B* **400**, 347 (1993); H.W. Hamber and R.M. Williams, *Nucl. Phys. B* **435**, 361 (1995); W. Beierl, E. Gerstenmayer, H. Markum, and J. Riedler, *Phys. Rev. D* **49**, 5231 (1994); *Nucl. Phys. B* **30** (Proc. Suppl.), 764 (1993); W. Beierl, B.A. Berg, B. Krishnan, H.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Markum, and J. Riedler, *Phys. Lett. B* **348**, 355 (1995); B.A. Berg and B. Krishnan, *Phys. Lett. B* **318**, 59 (1993).

[15] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995).

[16] H.W. Hamber, in *Proceedings of the 1984 Les Houches Summer School*, Session XLIII, Eds. K. Osterwalder and R. Stora (North Holland, Amsterdam, 1986).

[17] U. Wolff, *Phys. Rev. Lett.* **62**, 361 (1989); *Nucl. Phys. B* **322**, 759 (1989).

[18] A. Ferrenberg and R.H. Swendsen, *Phys. Rev. Lett.* **61**, 2635 (1988); *ibid.* **63**, 1658 (1989) (Erratum).

[19] C. Holm and W. Janke, *Phys. Lett. A* **173**, 8 (1993); *Phys. Rev. B* **48**, 936 (1993).

[20] K. Binder, *Z. Phys. B* **43**, 119 (1981).

[21] W. Janke *et al.*, *Phys. Lett. B* **315**, 412 (1993); *Phys. Rev. B* **49**, 9644 (1994).

[22] C. Holm and W. Janke, *Phys. Lett. B* **390**, 59 (1997); *Int. J. Mod. Phys. A* **14**, 3885 (1999).

[23] C. Holm and W. Janke, *Nucl. Phys. B* **47** (Proc. Suppl.), 465, 621 (1996); *Nucl. Phys. B* **477**, 465 (1996).

[24] P. Menotti and P.P. Peirano, *Phys. Lett. B* **353**, 444 (1995); J. Ambjørn, J.L. Nielsen, and J. Rolf, *Class. Quantum Grav.* **14**, 3225 (1997).

[25] H.W. Hamber and R.M. Williams, *Phys. Rev. D* **59**, 064014 (1999); H.W. Hamber, *Phys. Rev. D* **61**, 124008 (2000).

[26] J. Ambjørn and R. Loll, *Nucl. Phys. B* **536**, 407 (1998); J. Ambjørn, J. Correia, C. Kristjansen, and R. Loll, *Phys. Lett. B* **475**, 24 (2000).