Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

TIME-TRANSFORMATION APPROACH TO Q-DEFORMED OBJECTS

A. INOMATA

Department of Physics, SUNY-Albany, Albany, NY 12222, USA
E-mail: inomata@albany.edu

Time-transformations in path integrals are revisited. In particular the time-dependent coordinate transformation associated with the global time-transformation is discussed and applied for deriving the propagator of a q-deformed object.

1 Introduction

Duru and Kleinert [1] were the first to publish a paper on the time-transformation applied to a path integral. More specifically, they calculated a path integral for the hydrogen atom by using the time-transformation of Kustaanheimo and Stiefel [2] and succeeded for the first time in obtaining the Coulomb propagator from Feynman's path integral. Feynman asserted in his 1949 paper [3] by deriving Schrödinger's equation from the path integral that the path integral approach is equivalent to the standard canonical approach. However, the path integral remained incapable of solving the hydrogen atom problem until Duru and Kleinert made a breakthrough by bringing the time-transformation into the path-integral calculation.

After the success in the Coulomb problem, many authors have employed various time-transformations to solve other problems which can be solved by Schrödinger's equation but had not been solved by path integration [4]. Feynman's path-integral method, if time-transformations are appropriately used, is indeed capable of producing exact solutions for many integrable systems. In this manner, Feynman's approach is shown to be as powerful as the standard canonical approach in obtaining exact results. However, it is

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

as yet unclear whether the time-transformation technique is effectively applicable to other nontrivial problems. In this paper, we attempt to utilize the time-transformation technique in dealing with q-deformation.

Time-transformations, $t \to s$, pertinent to path integrals in nonrelativistic quantum mechanics, may be expressed in the form,

$$ds = F[\mathbf{r}(t), t]\, dt, \tag{1}$$

which may be classified into two types: global (holonomic) transformations and local (non-holonomic) transformations [5].

A holonomic time-transformation with $F = dg^{-1}/dt$ takes the time parameter t into a new "time" parameter s by

$$s = g^{-1}(t) \qquad \text{or} \qquad t = g(s). \tag{2}$$

In this case, the new parameter s is globally meaningful as the time parameter t is. An important example is the time-transformation considered by de Alfaro, Fubini, and Furlan [6], and by Jackiw [7] in combination with a time-dependent conformal coordinate transformation which will be discussed in Section 2.

A non-holonomic time-transformation for which Eq. (1) may be integrable along a certain path is generally significant only for an infinitesimal time interval. The time-transformation of Kustaanheimo and Stiefel [2], used for the path integral of the hydrogen atom, is a prototype of local time-transformations. The transformations used in carrying out path integration for exact results are mainly of the non-holonomic type.

In Section 2, we focus our attention on the global time-transformation in combination with a time-dependent conformal coordinate transformation. In Section 3 we use time-transformations to analyze q-deformed objects and derive the propagators of the q-deformed free particle and of the pulsed harmonic oscillator.

2 Time-Dependent Conformal Transformations

The time-dependent conformal transformation (TDCT) is an isotropic scale transformation of coordinates from $\mathbf{r}$ to $\mathbf{R}$:

$$\mathbf{r}(t) = \mathbf{R}(s)\, f(s), \tag{3}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

associated with a global time-transformation

$$t = g(s). \tag{4}$$

In the above we assume that $g(s)$ and $f(s)$ are both well-behaved single-valued functions of s and mutually related by

$$f^2(s) = \overset{\circ}{g}(s), \tag{5}$$

where $\overset{\circ}{g} = dg/ds$.

To see how TDCT works in a path integral [8], we consider a particle of charge e and mass M under the influence of a scalar potential $V(\mathbf{r})$ and a vector potential $\mathbf{A}(\mathbf{r})$ such that $\mathbf{A}(\mathbf{r}) \cdot \mathbf{r} = 0$. Upon application of TDCT, the action integral for the system,

$$S(t'', t') = \int_{t'}^{t''} \left[\frac{1}{2} M \dot{\mathbf{r}}^2 + \frac{e}{c} \mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} - V(\mathbf{r}) \right] dt, \tag{6}$$

transforms into

$$S(t'', t') = S_0(s'', s') + \tilde{S}(s'', s'), \tag{7}$$

where $s' = g^{-1}(t')$ and $s'' = g^{-1}(t'')$. The first term is a quantity depending only on the end point values,

$$S_0(s'', s') = \frac{1}{2} \left[M \mathbf{R}^2 \frac{\overset{\circ}{f}}{f} \right]_{s'}^{s''}, \tag{8}$$

which does not contribute to the equation of motion. The second term is the principal part of the new action,

$$\tilde{S}(s'', s') = \int_{s'}^{s''} \left[\frac{1}{2} M \overset{\circ}{\mathbf{R}}{}^2 + \frac{e}{c} \tilde{\mathbf{A}} \cdot \overset{\circ}{\mathbf{R}} - \tilde{V}(\mathbf{R}, s) \right] ds \tag{9}$$

with

$$\tilde{\mathbf{A}}(\mathbf{R}, s) = f(s) \mathbf{A}(f\mathbf{R}, g), \tag{10}$$

$$\tilde{V}(\mathbf{R}, s) = f^2(s)\, V(\mathbf{r}) + \frac{1}{2} M\, \omega^2(s)\, \mathbf{R}^2, \tag{11}$$

$$\omega^2(s) = (\overset{\circ\circ}{f}\, f - 2\, \overset{\circ}{f}{}^2)/f^2. \tag{12}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

If the original vector potential is given in the form $\mathbf{A}(\mathbf{r}) = \mathbf{a}(\theta, \phi)/r$ with $\mathbf{a} \cdot \mathbf{r} = 0$, the vector interaction term remains form-invariant: $\mathbf{A} \cdot d\mathbf{r} = \tilde{\mathbf{A}} \cdot d\mathbf{R}$. As is apparent from (11), TDCT, when applied to the scalar potential $V(\mathbf{r})$, generates an extra term which has the form of the time-dependent harmonic oscillator. Thus, for instance, if both the vector and scalar potentials are absent, TDCT converts the action of the free particle into that of a time-dependent harmonic oscillator. Or conversely, the inverse of TDCT reduces the harmonic oscillator to a free particle.

Next we take this transformation scheme to quantum mechanics via the path integral. The propagator corresponding to the action (6) is

$$K(\mathbf{r}'', t''; \mathbf{r}', t') = \int_{\mathbf{r}'=\mathbf{r}(t')}^{\mathbf{r}''=\mathbf{r}(t'')} e^{iS/\hbar}\, \mathcal{D}\mathbf{r}. \tag{13}$$

It can be shown explicitly by the time-sliced calculation [8] that TDCT changes this propagator into

$$K(\mathbf{r}'', t''; \mathbf{r}', t') = (f' f'')^{-3/2} \exp[iS_0(s'', s')/\hbar]\, \tilde{K}(\mathbf{R}'', s''; \mathbf{R}', s'), \tag{14}$$

where

$$\tilde{K}(\mathbf{R}'', s''; \mathbf{R}', s') = \int_{\mathbf{R}'=\mathbf{R}(s')}^{\mathbf{R}''=\mathbf{R}(s'')} e^{i\tilde{S}/\hbar}\, \mathcal{D}\mathbf{R}. \tag{15}$$

Therefore the transformed propagator can be calculated directly by substituting (3)-(5) as

$$\tilde{K}(\mathbf{R}'', s''; \mathbf{R}', s') = (f' f'')^{3/2} \exp[-iS_0(s'', s')/\hbar]\, K(\mathbf{r}'', t''; \mathbf{r}', t')|_{TDCT}. \tag{16}$$

As an example, let us take the Alfaro-Fubini-Furlen-Jackiw (AFFJ) transformation [6,7]:

$$\mathbf{r}(t) = \mathbf{R}(s) \sec(\omega s), \qquad t = \frac{\tan(\omega s)}{\omega}. \tag{17}$$

Apparently $f(s) = \sec(\omega s)$ and $g(s) = \omega^{-1} \tan(\omega s)$ satisfy the condition (5). Under this transformation, the action integral for a free particle ($\mathbf{A} = 0$ and $V = 0$) takes the form $S_0(s'', s') + \tilde{S}(s'', s')$ with

$$S_0(s'', s') = \frac{1}{2} M \left[\mathbf{R}^2(s'') - \mathbf{R}^2(s')\right] \tag{18}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

and

$$\tilde{S}(s'', s') = \int_{s'}^{s''} \left[\frac{1}{2}M\mathring{\mathbf{R}}^2 - \frac{1}{2}M\omega^2\mathbf{R}^2\right] ds. \tag{19}$$

The last action is the one for the simple harmonic oscillator. Since S_0 does not contribute to the equation of motion, the AFFJ transformation (17) converts the free particle into a harmonic oscillator [9]. The direct substitution of (17) via the formula (16) into the free particle propagator,

$$K^{(\text{free})}(\mathbf{r}'', t''; \mathbf{r}', t') = \left[\frac{M}{2\pi i\hbar(t''-t')}\right]^{3/2} \exp\left[\frac{iM(\mathbf{r}''-\mathbf{r}')^2}{2\hbar(t''-t')}\right], \tag{20}$$

leads to the propagator of the harmonic oscillator,

$$\tilde{K}^{(\text{osc})}(\mathbf{R}'', s''; \mathbf{R}', s') = \left[\frac{M\omega}{2\pi i\hbar\, \sin[\omega(s''-s')]}\right]^{3/2}$$
$$\times \exp\left[\frac{iM\omega}{2\hbar\sin[\omega(s''-s')]}\left\{(\mathbf{R}'^2 + \mathbf{R}''^2)\cos[\omega(s''-s')] - 2\mathbf{R}'\mathbf{R}''\right\}\right]. \tag{21}$$

In this process, we have utilized the following identities [9],

$$\frac{A\,\sec^2\alpha + B\,\sec^2\beta}{\tan\alpha - \tan\beta} = (A+B)\,\cot(\alpha-\beta) + A\,\tan\alpha - B\,\tan\beta, \tag{22}$$

$$\frac{\sec\alpha\,\sec\beta}{\tan\alpha - \tan\beta} = \csc(\alpha-\beta). \tag{23}$$

Other examples can be found in Refs. [8,10].

3 Global Time-Transformation in q-Deformation

In this section, we study global time-transformations related to q-deformation.

3.1 The q-Free Particle

Let us start with the Newtonian free particle in one dimension which obeys the difference equation

$$x(t+T) - 2x(t) + x(t-T) = 0, \tag{24}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

where T is any finite-time period. Obviously its solution is given by

$$x(t) = at + b, \tag{25}$$

where a and b are constants.

Now we apply to the free particle the following transformation [11],

$$x(t) \to y(\tau) = x(\tau), \qquad t \to \tau = q^{2t/T}, \tag{26}$$

where $q \neq 0$ and $q \neq 1$. The new variable,

$$y(\tau(t)) = a\tau + b = aq^{2t/T} + b, \tag{27}$$

does not satisfy (24) any longer but obeys the equation

$$q^{-1}y(q^2\tau) - (q + q^{-1})y(\tau) + qy(q^{-2}\tau) = 0. \tag{28}$$

While the difference equation (24) dictates the time-evolution of $x(t)$ under the discrete time-translation $t - T \to t \to t + T$, the new difference equation (28) stipulates the progression of $y(\tau)$ under the time-scaling $q^{-2}\tau \to \tau \to q^2\tau$.

In fact, the difference equation (28) is equivalent to the q-deformation counterpart of the force-free Newton equation:

$$\frac{D^2 y(\tau)}{D\tau^2} = 0, \tag{29}$$

where $Dy(\tau)/D\tau$ is the symmetric Jackson q-derivative defined by [12]

$$\frac{Dy(\tau)}{D\tau} = \frac{y(q\tau) - y(q^{-1}\tau)}{(q - q^{-1})\tau}. \tag{30}$$

Thus the scaling factor q appearing in Eq. (26) turns out to be the same as the deformation parameter.

In this way, the time-transformation (26) converts the Newtonian free particle to the q-deformed object obeying Eq. (28) or Eq. (29) which we call the q-deformed free particle or the q-free particle in short.

In order for the new time parameter τ to be real for any t, the deformation parameter q must be a positive real number. In this case, the trajectory described by (27) is also real and continuous. Thus the q-free particle moving along a real continuous trajectory is characterized by a positive real parameter $q \in \mathbf{R}^+$. If, however, the object is allowed only to hop along a real discrete sequential trajectory $\{y(s(nT))\}$ associated with discrete periodic

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

time-translations $t = nT$ $(n \in \mathbf{N})$, then q may be negative. Thus, for the proper q-free particle and the hopping q-free particle we have $q \in \mathbf{R}/(0,1)$.

If we let $x_n = x(t_0 + nT)$ $(n \in \mathbf{N}_0)$, then Eq. (24) may be written as the recursion relation

$$x_{n+1} - 2x_n + x_{n-1} = 0. \tag{31}$$

Correspondingly, defining $\tau_n = q_{2n}\tau_0$ with $\tau_0 = q^{2t_0/T}$, we obtain the recursion relation obeyed by $y_n = y(\tau_n)$:

$$q^{-1}y_{n+1} - (q + q^{-1})y_n + qy_{n-1} = 0. \tag{32}$$

3.2 The q-Object

Next we apply the time-dependent coordinate transformation

$$y(\tau) \;\rightarrow\; Q(\tau) = \tau^{-1/2}y(\tau) \tag{33}$$

to the q-free particle. Under this scale transformation, the difference equation (28) becomes

$$Q(q^2\tau) - (q + q^{-1})Q(\tau) + Q(q^{-2}\tau) = 0. \tag{34}$$

As is evident from Eqs. (27) and (33), the solution for this equation is given by

$$Q(\tau) = a\tau^{1/2} + b\tau^{-1/2}, \tag{35}$$

which may be rewritten as

$$Q(\tau) = \left(A\,\frac{\tau^{1/2} - \tau^{-1/2}}{\tau^{1/2} + \tau^{-1/2}} + B\right)(\tau^{1/2} + \tau^{-1/2}), \tag{36}$$

where $A = (a - b)/2$ and $B = (a + b)/2$.

Then it becomes clear that $Q(\tau)$ may be related to the coordinate variable $x(t)$ of the Newtonian free particle via a time-dependent scale transformation $x(t) \rightarrow Q(s)$ associated with the time-transformation $t \rightarrow s \in \mathbf{R}$:

$$x(t) = 2[(c/T)\ln q]^{1/2}Q(s)(q^{s/T} + q^{-s/T})^{-1}, \quad t = c\frac{q^{s/T} - q^{-s/T}}{q^{s/T} + q^{-s/T}}, \tag{37}$$

where

$$Q(s) = Q(\tau(s)) = \left(A\,\frac{q^{s/T} - q^{-s/T}}{q^{s/T} + q^{-s/T}} + B\right)(q^{s/T} + q^{-s/T}), \tag{38}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

and c is a constant to be chosen such that the condition (5) is met.

Although we have assumed $q \neq 1$ in Eq. (26), we may remove the assumption if we define $Q(s)$ by Eq. (37). It is obvious from Eq. (37) that $s \to t$ and $Q(s) \to x(t)$ in the limit $q \to 1$. The difference equation (34) also reduces to the equation for the free particle (24) when $q \to 1$. Furthermore, Eq. (34) indicates that the function $Q(s)$ may remain to be real-valued under the condition

$$q + q^{-1} \in \mathbf{R}. \tag{39}$$

This condition implies either $q \in \mathbf{R}$ or $q \in S^1$. In this regard, the object described by the real-valued variable $Q(s)$ is more general than the q-free particle. We refer to this generic object subjected to the condition (39) as the q-object.

Let $s_n = s_0 + nT$ and $Q_n = Q(s_n)$ where $n \in \mathbf{N}_0$. Then it is evident that Q_n satisfies the relation,

$$Q_{n+1} - (q + q^{-1})Q_n + Q_{n-1} = 0, \tag{40}$$

which is nothing but the recursion relation obeyed by the Chebyshev polynomials,

$$T_n(\cos\varphi) = \cos(n\varphi) \quad \text{and} \quad U_n(\cos\varphi) = \sin(n\varphi), \tag{41}$$

with $q = e^{-i\varphi}$ ($\varphi \in \mathbf{C}$). Thus the time-evolution of the q-object is a real Chebyshev process (with $q + q^{-1} \in \mathbf{R}$).

3.3 The Number Representation

The deformed harmonic oscillator of Biedenharn [13] and Macfarlane [14] is a well-known q-deformed object based on the commutator

$$\hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = q^{-\hat{N}}. \tag{42}$$

In the Fock space $\mathcal{F} = \{|n\rangle : n \in \mathbf{N}_0\}$, the operators $\hat{N}$, $\hat{a}^\dagger\hat{a}$ and $\hat{a}\hat{a}^\dagger$ are diagonalized with the diagonal elements, n, u_n and u_{n+1}, respectively; $\hat{a}^\dagger$ and $\hat{a}$ shift the energy states as

$$\hat{a}^\dagger|n\rangle = \sqrt{u_{n+1}}\,|n+1\rangle, \qquad \hat{a}|n\rangle = \sqrt{u_n}\,|n-1\rangle. \tag{43}$$

Here u_n is often called the structure function of the algebra (42).

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

By using Eq. (42) it is straightforward to show that the structure function u_n satisfies the recursion relation

$$u_{n+1} - (q + q^{-1})u_n + u_{n-1} = 0. \tag{44}$$

Surprisingly this recursion relation coincides with Eq. (40). However, we do not hastily conclude that the q-object is a q-deformed harmonic oscillator. The deformed oscillator is characterized by a Hamiltonian given as a function of $\hat{a}$ and $\hat{a}^\dagger$, whereas the q-object has nothing to do with a Hamiltonian. Here, associating $|n\rangle$ with the position of the q-object at a time $t = t_0 + nT$, we just point out that the discrete time-evolution of the q-object can be represented by the transition of the Fock states.

3.4 The p-Oscillator

The pulsed harmonic oscillator (p-oscillator) is defined here as a free particle which undergoes periodic pulses of Hooke's force $F(t) = -M\omega^2 X\delta(t/T - m)$, where $M\omega^2$ is Hooke's constant, T is the period of pulses and $m \in \mathbf{Z}$. The symmetrized action of this system for a time interval containing a single pulse is given by

$$S(t_m, t_{m-1}) = \frac{M}{2T}(X_m - X_{m-1})^2 - \frac{1}{4}M\omega^2 T\,(X_m^2 + X_{m-1}^2). \tag{45}$$

Calculating the canonical momenta by

$$\begin{aligned} P_m &= \partial S/\partial X_m &&= (M/T)(X_m - X_{m-1}) - (M\omega^2 T/2)X_m, \\ P_{m-1} &= -\partial S/\partial X_{m-1} &&= (M/T)(X_m - X_{m-1}) + (M\omega^2 T/2)X_{m-1}, \end{aligned} \tag{46}$$

we find the area-preserving linear map in phase space:

$$\begin{aligned} X_m &= X_{m-1} + (T/2M)(1 - \omega^2 T^2/4)^{-1}(P_m + P_{m-1}), \\ P_m &= P_{m-1} - (M\omega^2 T/2)(X_m + X_{m-1}). \end{aligned} \tag{47}$$

It is interesting to notice that both X_m and P_m satisfy the recursion relation

$$X_{m+1} - (2 - \omega^2 T^2)X_m + X_{m-1} = 0. \tag{48}$$

Obviously the time-evolution of the p-oscillator is also a Chebyshev process with

$$\varphi = \cos^{-1}(1 - \omega^2 T^2/2). \tag{49}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

If $0 < \omega^2 T^2 < 4$, then $\varphi \in \mathbf{R}$ and the discrete trajectory $\{X_m\}$ oscillates sinusoidally. If $\omega^2 T^2 < 0$ or $4 < \omega^2 T^2$, then φ is a complex number, and the solutions are no longer physical and do not represent the p-oscillator. Therefore the proper p-oscillator is a q-object with $q \in S^1$, i.e. $q = e^{-i\varphi}$ ($\varphi \in \mathbf{R}$).

4 The Propagator for the *q*-Object

Here we utilize the time-dependent scale transformation (37) to derive the propagator for the q-object and discuss special cases.

4.1 The q-Object

Letting $c = iT/\varphi$, we put the transformation (37) into the form,

$$x_n = Q_n \sec(in \ln q), \qquad t_n = (iT/\ln q)\tan(in \ln q). \tag{50}$$

Then, making use of the identities (22) and (23) once again, we can implement the transformation (50) to the one-dimensional free particle propagator for the time interval $t_n - t_0 = nT$,

$$K^{(\text{free})}(x_n, x_0; nT) = \left[\frac{M}{2\pi i\hbar(t_n - t_0)}\right]^{1/2} \exp\left[\frac{iM}{2\hbar}\frac{(x_n - x_0)^2}{t_n - t_0}\right], \tag{51}$$

to find the propagator for the q-object

$$\begin{aligned}\hat{K}^{(q-\text{object})}(Q_n, Q_0; nT) &= \left[\frac{M(q - q^{-1})}{2\pi i\hbar T(q^n - q^{-n})}\right]^{1/2} \\ &\times \exp\left[\frac{iM(q - q^{-1})}{4\hbar T(q^n - q^{-n})}\{(Q_n^2 + Q_0^2)(q^n + q^{-n}) - 4Q_n Q_0\}\right],\end{aligned} \tag{52}$$

where $q + q^{-1} \in \mathbf{R}$.

4.2 The p-Oscillator

For $q = e^{-i\varphi} \in S^1$ with $\varphi = \cos^{-1}(1 - \omega^2 T^2/2)$, we let $Q_n = X_n \in \mathbf{R}$ in Eq. (52). Then the propagator for the proper p-oscillator is given by

$$\hat{K}^{(\text{p}-\text{osc})}(X_n, X_0; nT) = \left[\frac{M\, U_1(\cos\varphi)}{2\pi i\hbar T\, U_n(\cos\varphi)}\right]^{1/2}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

$$\times \exp\left[\frac{iM\,U_1(\cos\varphi)}{2\hbar T U_n(\cos\varphi)}\{(X_n^2+X_0^2)T_n(\cos\varphi)-2X_nX_0\}\right], \tag{53}$$

where $T_n(\cos\varphi)$ and $U_n(\cos\varphi)$ are the Chebyshev polynomials given in Eq. (41). The caustics of the propagator correspond to $q^n = e^{im\pi}$ $(m \in \mathbf{Z})$. In the limit $T \to 0$ $(q \to 1)$ and $n \to \infty$, such that the total time interval nT remains constant, Eq. (53) approaches the propagator for the usual harmonic oscillator.

4.3 The q-Free Particle

The propagator for the T-evolution of the q-free particle is obtained by transforming X_n of Eq. (52) back to $y_n = q^n X_n$ via Eq. (50). Namely,

$$\hat{K}^{(q-\text{free})}(y_n, y_0; nT) = \left[\frac{M(q-q^{-1})}{2\pi i\hbar T(q^n-q^{-n})}\right]^{1/2}$$
$$\times \exp\left[\frac{iM(q-q^{-1})}{4\hbar T(q^n-q^{-n})}\{(q^{-2n}y_n^2+y_0^2)(q^n+q^{-n})-4q^{-n}y_ny_0\}\right]. \tag{54}$$

For the proper q-free particle, $q \in \mathbf{R}^+$. If we include the hopping type in the q-free particle, then q can take any real number excluding $q = 0$. In the limit $q \to 1$, Eq. (55) reduces to the propagator for the usual free particle.

5 Concluding Remarks

The global time-transformations have been used to discuss q-deformed objects. The propagator for the generic q-object is also obtained, which contains those of the pulsed harmonic oscillator and the q-free particle as special cases. Other aspects of the path integral for the q-object will be given elsewhere [15].

It is an interesting question why the AFFJ transformation (17) can generate the discrete energy spectrum for the harmonic oscillator out of the continuous spectrum of the free particle. By the q-analysis, it becomes clear that the AFFJ transformation in the q-version (50) is a sort of analytical continuation through q. As the bound and the continuous states of the Coulomb problem are related to the compact group $O(4)$ and the non-compact group $O(3,1)$, respectively, the q-free particle with $q \in \mathbf{R}^+$ is analytically continued to the p-oscillator with $q \in S^1$. The generation of the discrete spectrum can be ascribed to the compactification of the space belonging to the deformation parameter q.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

References

[1] I.H. Duru and H. Kleinert, *Phys. Lett. B* **84**, 185 (1979); *Fortschr. Phys.* **30**, 401 (1982). For the time-sliced path integration, see R. Ho and A. Inomata, *Phys. Rev. Lett.* **48**, 231 (1982).

[2] P. Kustaanheimo and E. Stiefel, *J. Rein. Angew. Math.* **218**, 204 (1965).

[3] R.P. Feynman, *Rev. Mod. Phys.* **76**, 769 (1949).

[4] See, for instance, H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995); A. Inomata, H. Kuratsuji, and C.C. Gerry, *Path Integrals and Coherent States of* $SU(2)$ *and* $SU(1,1)$ (World Scientific, Singapore, 1992); C. Grosche and F. Steiner, *Handbook of Feynman Path Integrals* (Springer-Verlag, Berlin, 1998).

[5] A. Inomata, in *Path Summation: Achievements and Goals*, Eds. S. Lundviquist, A. Rangfani, V. Sayakanit, and L.S. Schulman (World Scientific, Singapore, 1988), p.114. For a unification of the two types, see A. Pelster and A. Wunderlin, *Z. Phys. B* **89**, 373 (1992).

[6] V. de Alfaro, S. Fubini, and G. Furlan, *Nuovo Cim.* **34**, 569 (1976).

[7] R. Jackiw, *Ann. Phys. (N.Y.)* **129**, 183 (1980).

[8] J.M. Cai, P.Y. Cai, and A. Inomata, *Proceedings of International Symposium on Advanced Topics of Quantum Physics - Shanxi'92*, Eds. J.Q. Liang, M.L. Wang, S.N. Qiao, and D.C. Su (Science Press, Beijing, 1993), p. 75.

[9] P.Y. Cai, A. Inomata, and P. Wang, *Phys. Lett. A* **91**, 331 (1982).

[10] G. Junker and A. Inomata, *Phys. Lett. A* **110**, 195 (1985).

[11] A. Dimakis and F. Müller-Hoissen, *Phys. Lett. B* **295**, 242 (1992).

[12] See, e.g. M. Chainchian and A. Domichev, *Introduction to Quantum Groups* (World Scientific, Singapore, 1996), p. 154.

[13] L.C. Biedenharn, *J. Phys. A* **22**, L873 (1989).

[14] A.J. Macfarlane, *J. Phys. A* **22**, 4581 (1989).

[15] A. Inomata, J.C. Kimball, and C. Pigorsch, SUNY-A preprint (2000).