Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

RECURSIVE CONSTRUCTION OF FEYNMAN GRAPHS IN SPONTANEOUSLY BROKEN $O(N)$-SYMMETRIC ϕ^4-THEORY

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We consider $O(N)$-symmetric ϕ^4-theory in its spontaneously broken phase and investigate how the corresponding one-particle irreducible Feynman graphs with arbitrary numbers of external legs can recursively be constructed. In particular, we sketch the derivation of the necessary identities for the effective energy Γ (or, with trivial modifications, effective action), which subsequently can be converted into recursion relation for graphs. Although the resulting relations will not be as concise as their counterpart in the $N = 1$ case, they nevertheless should be a useful means for graph generation once implemented on a computer.

First let me note that it is a pleasure for me to contribute to the anniversary edition in honor of Prof. Hagen Kleinert. Let this be a small but original contribution to one of the many areas of theoretical physics he is and he has been working on through the course of the years. The article can be viewed as a direct formalized consequence of his work in Refs. [1,2].

The $O(N)$-symmetric ϕ^4-theory is a field theoretic model which serves not only as a toy model in particle physics, but also covers an important subset of universality classes in the context of critical phenomena. While several universal quantities can be derived upon working in the symmetric (or disordered) phase, there are a number of amplitude ratios whose calculation necessitates to consider the spontaneously broken (or ordered) phase. This results in the context of perturbation theory in a vast increase of diagrams to be considered even in low loop orders. Part of this increase is due to three-point vertices arising in the ordered phase, which also happens already in the

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$N = 1$ case. The other part of the increase is due to the presence of two different propagators instead of only one in the $N = 1$ case. For a classic reference, see Ref. [3]. A more recent reference is Ref. [4], which also contains a list of related references.

In the context of critical phenomena (and not only there), we are mainly interested in one-particle irreducible (1PI) diagrams. There are several ways to obtain these diagrams for $N > 1$ (for $N = 1$, see Ref. [5] or, for an alternative approach, see Ref. [6]). Among them are:

(1) Use the appropriate Feynman rules.
(2) First derive the graphs for single-component ϕ^4-theory, then replace each propagator by a sum of Higgs and Goldstone propagators and throw away those graphs that are not permitted.
(3) Write down recursion relations for the connected graphs and throw away all non-1PI graphs.
(4) Introduce a mixed propagator which is Higgs at one end and Goldstone at the other and translate the recursion relations for the single-component theory into recursion relations for the case at hand. Set the mixed propagator to zero in the resulting diagrams, so only permitted diagrams survive.
(5) Develop recursion relations for the broken-symmetry 1PI graphs.

Strategy 1 is fraught with the danger of making mistakes, both by hand or when programming. Strategies 2-4 produce a huge number of diagrams at intermediate stages that are going to be dropped eventually. Strategy 5 is the one suggested here. It certainly results in rather elaborate equations, as we will see below. However, once everything is automated, it appears to be superior to the other methods. This is especially true if even the generation of the necessary equations is automated, a task left for future work, since in this brief report, we present equations derived "by hand".

Straightforwardly generalizing earlier work in Refs. [5,7] (we use conventions introduced there), let the energy be given by

$$
\begin{aligned}
E[\phi,\chi,J,I,H,G] = C &+ \int_1 (J_1-\hat{J}_1)\phi_1 + \int_1 (I_1-\hat{I}_1)\chi_1 \\
&+\frac{1}{2}\int_{12} H^{-1}_{12}\phi_1\phi_2+\frac{1}{2}\int_{12} G^{-1}_{12}\chi_1\chi_2+\frac{1}{6}\int_{123} S_{123}\phi_1\phi_2\phi_3+\frac{1}{2}\int_{123} T_{123}\phi_1\chi_2\chi_3 \\
&+\frac{1}{24}\int_{1234} L_{1234}\phi_1\phi_2\phi_3\phi_4+\frac{1}{4}\int_{1234} M_{1234}\phi_1\phi_2\chi_3\chi_4+\frac{1}{24}\int_{1234} N_{1234}\chi_1\chi_2\chi_3\chi_4
\end{aligned}
\tag{1}
$$

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with appropriately symmetrized bare propagators H and G and interactions S, T, L, M, and N. The indices represent real-space arguments as well as group indices, while the integration signs stand not only for space integrations, but also for summations over group indices. The currents $\hat{J}$ and $\hat{I}$ will be used to define Legendre transforms while keeping the option of having non-zero currents J and I in the effective energy.

Define the functionals Z and W by

$$Z[\hat{J},\hat{I},H,G] = \exp W[\hat{J},\hat{I},H,G] = \int D\phi D\chi \exp(-E[\phi,\chi,\hat{J},\hat{I},H,G]). \quad (2)$$

There is an infinite set of identities we can derive for W. A few of the simplest ones appear particularly useful for our purposes. They are

$$\begin{aligned}
0 &= e^{-W[\hat{J},\hat{I},H,G]} \int D\phi D\chi \frac{\delta}{\delta\phi_1} e^{-E[\phi,\chi,\hat{J},\hat{I},H,G]} \\
&= \hat{J}_1 - J_1 - \int_2 H_{12}^{-1} \frac{\delta W}{\delta \hat{J}_2} + \int_{23} S_{123} \frac{\delta W}{\delta H_{23}^{-1}} + \int_{23} T_{123} \frac{\delta W}{\delta G_{23}^{-1}} \\
&\quad + \frac{1}{3} \int_{234} L_{1234} \left(\frac{\delta^2 W}{\delta \hat{J}_2 \delta H_{34}^{-1}} + \frac{\delta W}{\delta \hat{J}_2} \frac{\delta W}{\delta H_{34}^{-1}} \right) \\
&\quad + \int_{234} M_{1234} \left(\frac{\delta^2 W}{\delta \hat{J}_2 \delta G_{34}^{-1}} + \frac{\delta W}{\delta \hat{J}_2} \frac{\delta W}{\delta G_{34}^{-1}} \right),
\end{aligned} \quad (3)$$

$$\begin{aligned}
0 &= e^{-W[\hat{J},\hat{I},H,G]} \int D\phi D\chi \frac{\delta}{\delta\phi_1} \left(\phi_2 e^{-E[\phi,\chi,\hat{J},\hat{I},H,G]} \right) \\
&= \delta_{12} + (\hat{J}_1 - J_1) \frac{\delta W}{\delta \hat{J}_2} + 2 \int_3 H_{13}^{-1} \frac{\delta W}{\delta H_{23}^{-1}} \\
&\quad + \int_{34} S_{134} \left(\frac{\delta^2 W}{\delta \hat{J}_2 \delta H_{34}^{-1}} + \frac{\delta W}{\delta \hat{J}_2} \frac{\delta W}{\delta H_{34}^{-1}} \right) \\
&\quad + \int_{34} T_{134} \left(\frac{\delta^2 W}{\delta \hat{J}_2 \delta G_{34}^{-1}} + \frac{\delta W}{\delta \hat{J}_2} \frac{\delta W}{\delta G_{34}^{-1}} \right) \\
&\quad - \frac{2}{3} \int_{345} L_{1345} \left(\frac{\delta^2 W}{\delta H_{23}^{-1} \delta H_{45}^{-1}} + \frac{\delta W}{\delta H_{23}^{-1}} \frac{\delta W}{\delta H_{45}^{-1}} \right) \\
&\quad - 2 \int_{345} M_{1345} \left(\frac{\delta^2 W}{\delta H_{23}^{-1} \delta G_{45}^{-1}} + \frac{\delta W}{\delta H_{23}^{-1}} \frac{\delta W}{\delta G_{45}^{-1}} \right),
\end{aligned} \quad (4)$$

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$$
\begin{aligned}
0 &= e^{-W[jh,\hat{I},H,G]}\int D\phi D\chi \frac{\delta}{\delta\chi_1}\left(\chi_2 e^{-E[\phi,\chi,\hat{J},\hat{I},H,G]}\right)\\
&= \delta_{12} + (\hat{I}_1 - I_1)\frac{\delta W}{\delta \hat{I}_2} + 2\int_3 G^{-1}_{13}\frac{\delta W}{\delta G^{-1}_{23}}\\
&\quad +2\int_{34} T_{134}\left(\frac{\delta^2 W}{\delta\hat{J}_3\delta G^{-1}_{24}} + \frac{\delta W}{\delta \hat{J}_3}\frac{\delta W}{\delta G^{-1}_{24}}\right)\\
&\quad -2\int_{345} M_{4513}\left(\frac{\delta^2 W}{\delta G^{-1}_{23}\delta H^{-1}_{45}} + \frac{\delta W}{\delta G^{-1}_{23}}\frac{\delta W}{\delta H^{-1}_{45}}\right)\\
&\quad -\frac{2}{3}\int_{345} N_{1345}\left(\frac{\delta^2 W}{\delta G^{-1}_{23}\delta G^{-1}_{45}} + \frac{\delta W}{\delta G^{-1}_{23}}\frac{\delta W}{\delta G^{-1}_{45}}\right). \qquad (5)
\end{aligned}
$$

Next, define W_0 and W_I by

$$W_0 = W_{S,T,L,M,N=0}, \qquad W = W_0 + W_I. \qquad (6)$$

For W_0, the identities (3)-(5) reduce to

$$\frac{\delta W_0}{\delta J_1} = \int_2 H_{12}J_2, \qquad \frac{\delta W_0}{\delta I_1} = \int_2 G_{12}I_2, \qquad (7)$$

$$\frac{\delta W_0}{\delta H^{-1}_{12}} = -\frac{1}{2}\left(H_{12} + \int_3 H_{13}J_3\int_4 H_{24}J_4\right), \qquad (8)$$

$$\frac{\delta W_0}{\delta G^{-1}_{12}} = -\frac{1}{2}\left(G_{12} + \int_3 G_{13}I_3\int_4 G_{24}I_4\right). \qquad (9)$$

With appropriate normalization of the path integration measure, this is solved by

$$
\begin{aligned}
W_0 &= -C - \frac{1}{2}\int_1 (\ln H^{-1})_{11} - \frac{1}{2}\int_1 (\ln G^{-1})_{11}\\
&\quad +\frac{1}{2}\int_{12} H_{12}(J_1-\hat{J}_1)(J_2-\hat{J}_2) + \frac{1}{2}\int_{12} G_{12}(I_1-\hat{I}_1)(I_2-\hat{I}_2). \qquad (10)
\end{aligned}
$$

Now the identities (3)-(5) may be translated into identities for W_I and then into recursion relations for the connected graphs of the theory. One possibility to obtain the 1PI graphs would be to just throw away the non-1PI graphs. However, this would generate a vast number of connected, but non-1PI graphs to be thrown away later anyway and therefore will very soon be an exploding task. Instead, let us right away turn to the effective energy, the generating functional of 1PI graphs.

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Define the effective energy Γ by a twofold Legendre transformation,

$$\Gamma[\sigma, \pi, H, G] = -W[\hat{J}, \hat{I}, H, G] + \int_1 \hat{J}_1 \sigma_1 + \int_1 \hat{I}_1 \pi_1, \tag{11}$$

with new independent variables

$$\sigma_1 = \left(\frac{\delta W}{\delta \hat{J}_1}\right)_{\hat{I}HG}, \qquad \pi_1 = \left(\frac{\delta W}{\delta \hat{I}_1}\right)_{\hat{J}HG}. \tag{12}$$

As usual,

$$\left(\frac{\delta \Gamma}{\delta \sigma_1}\right)_{\pi HG} = \hat{J}_1, \qquad \left(\frac{\delta \Gamma}{\delta \pi_1}\right)_{\sigma HG} = \hat{I}_1. \tag{13}$$

Define further Γ_0 and Γ_I by

$$\Gamma_0 = \Gamma_{S,T,L,M,N=0}, \qquad \Gamma = \Gamma_0 + \Gamma_I. \tag{14}$$

With

$$\sigma_1 = \frac{\delta W_0}{\delta \hat{J}_1} = \int_2 H_{12}(\hat{J}_2 - J_2), \qquad \pi_1 = \frac{\delta W_0}{\delta \hat{I}_1} = \int_2 G_{12}(\hat{I}_2 - I_2) \tag{15}$$

we have

$$\hat{J}_1 = J_1 + \int_2 H_{12}^{-1} \sigma_2, \qquad \hat{I}_1 = I_1 + \int_2 G_{12}^{-1} \pi_2, \tag{16}$$

and therefore with (10)

$$\begin{aligned} \Gamma_0 = C &+ \frac{1}{2}\int_1 (\ln H^{-1})_{11} + \frac{1}{2}\int_1 (\ln G^{-1})_{11} + \int_1 J_1 \sigma_1 + \int_1 I_1 \pi_1 \\ &+ \frac{1}{2}\int_{12} H_{12}^{-1} \sigma_1 \sigma_2 + \frac{1}{2}\int_{12} G_{12}^{-1} \pi_1 \pi_2. \end{aligned} \tag{17}$$

This is the only place in Γ, where J and I still appear, in other words: Γ_I is independent of J and I.

Now we can translate (3)-(5) into identities for Γ_I, which in turn can be converted into recursion relations for Feynman graphs. The main complication arises from the fact that we now deal with a twofold Legendre transform, which destroys some of the ease we are used to from dealing with the $N = 1$ case [5], since the relations between the second derivatives of W and Γ become more complicated. Instead of translating (3)-(5) in full generality, we use for each situation the simplest identity available.

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Let us for the remainder invoke a generalized Einstein summation convention, where we sum/integrate over the variables represented by repeated indices. For the generation of diagrams with at least one external field σ, we translate the simplest identity (3) and obtain

$$\begin{aligned}
0 &= \frac{\delta\Gamma_I}{\delta\sigma_1}\sigma_1 - \frac{1}{2}S_{123}\sigma_1 H_{23} - \frac{1}{2}S_{123}\sigma_1\sigma_2\sigma_3 - \frac{1}{2}T_{123}\sigma_1 G_{23} - \frac{1}{2}T_{123}\sigma_1\pi_2\pi_3 \\
&-\frac{1}{2}L_{1234}\sigma_1\sigma_2 H_{34} - \frac{1}{6}L_{1234}\sigma_1\sigma_2\sigma_3\sigma_4 - \frac{1}{2}M_{1234}\sigma_1\sigma_2 G_{34} - \frac{1}{2}M_{1234}\sigma_1\sigma_2\pi_3\pi_4 \\
&+S_{123}\sigma_1 H_{24}H_{35}\frac{\delta\Gamma_I}{\delta H_{45}} + T_{123}\sigma_1 G_{24}G_{35}\frac{\delta\Gamma_I}{\delta G_{45}} \\
&+L_{1234}\sigma_1\sigma_2 H_{35}H_{46}\frac{\delta\Gamma_I}{\delta H_{56}} + L_{1234}\sigma_1 H_{25}H_{36}H_{47}\Gamma^{\sigma H}_{657} \\
&+2M_{1234}\sigma_1\pi_3 H_{25}G_{46}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta\pi_6} + 2M_{1234}\sigma_1\pi_3 H_{25}G_{46}\Gamma^{G}_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta\pi_8} \\
&+2M_{1234}\sigma_1\pi_3 H_{25}\frac{\delta\Gamma_I}{\delta H_{56}}H_{67}G_{48}\frac{\delta^2\Gamma_I}{\delta\sigma_7\delta\pi_8} \\
&+2M_{1234}\sigma_1\pi_3 H_{25}\frac{\delta\Gamma_I}{\delta H_{56}}H_{67}G_{48}\Gamma_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_7\delta\pi_0} \\
&+M_{1234}\sigma_1\sigma_2 G_{35}G_{46}\frac{\delta\Gamma_I}{\delta G_{56}} + M_{1234}\sigma_1 H_{25}G_{36}G_{47}\Gamma^{\sigma G}_{657}. \qquad (18)
\end{aligned}$$

For the generation of diagrams with no Higgs fields σ and no Higgs propagator H, we may translate (5) with $T, M = 0$ and obtain

$$\begin{aligned}
0 &= \frac{\delta\Gamma_I}{\delta\pi_1}\pi_1 + 2\frac{\delta\Gamma_I}{\delta G_{12}}G_{12} - \frac{1}{6}N_{1234}\pi_1\pi_2\pi_3\pi_4 - N_{1234}G_{12}\pi_3\pi_4 - \frac{1}{2}N_{1234}G_{12}G_{34} \\
&+2N_{123456}\pi_1\pi_2 G_{35}G_{46}\frac{\delta\Gamma_I}{\delta G_{56}} + 2N_{1234}G_{12}G_{35}G_{46}\frac{\delta\Gamma_I}{\delta G_{56}} \\
&+\frac{4}{3}N_{1234}\pi_1 G_{25}G_{36}G_{47}\frac{\delta^2\Gamma_I}{\delta\pi_5\delta G_{67}} + \frac{2}{3}N_{1234}G_{15}G_{26}G_{37}G_{48}\frac{\delta^2\Gamma_I}{\delta G_{56}\delta G_{78}} \\
&-\frac{2}{3}N_{1234}G_{15}G_{26}\frac{\delta\Gamma_I}{\delta G_{56}}G_{37}G_{48}\frac{\delta\Gamma_I}{\delta G_{78}} - \frac{8}{3}N_{1234}\pi_1 G_{25}\frac{\delta\Gamma_I}{\delta G_{56}}G_{67}\frac{\delta^2\Gamma_I}{\delta\pi_7\delta G_{89}}G_{83}G_{94} \\
&-\frac{2}{3}N_{1234}G_{15}G_{26}\frac{\delta^2\Gamma_I}{\delta G_{56}\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta G_{90}}G_{93}G_{04} \\
&+\frac{4}{3}N_{1234}G_{15}G_{26}\frac{\delta^2\Gamma_I}{\delta G_{56}\delta\pi_7}G_{78}\frac{\delta\Gamma_I}{\delta G_{89}}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta G_{\bar{1}\bar{2}}}G_{\bar{1}3}G_{\bar{2}4}, \qquad (19)
\end{aligned}$$

where it is understood that in all terms we have set $T, M = 0$.

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For the generation of all other 1PI diagrams, i.e. diagrams with no Higgs field σ, but at least one Higgs propagator H, we translate (4) with $\sigma = 0$ and obtain

$$
\begin{aligned}
0 = {} & 2\frac{\delta\Gamma_I}{\delta H_{12}}H_{12} + 2T_{123}H_{14}\frac{\delta^2\Gamma_I}{\delta\sigma_4\delta\pi_5}G_{52}\pi_3 + 2T_{123}H_{14}\frac{\delta^2\Gamma_I}{\delta\sigma_4\delta\pi_5}G_{56}\Gamma^G_{67}G_{72}\pi_3 \\
& -4T_{123}H_{14}\frac{\delta\Gamma_I}{\delta H_{45}}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta\pi_7}G_{72}\pi_3 - 4T_{123}H_{14}\frac{\delta\Gamma_I}{\delta H_{45}}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta\pi_7}G_{78}\Gamma^G_{89}G_{92}\pi_3 \\
& +S_{123}H_{14}H_{25}H_{36}\Gamma^{\sigma H}_{456} + T_{123}H_{14}G_{25}G_{36}\Gamma^{\sigma G}_{456} - \frac{1}{2}L_{1234}H_{12}H_{34} \\
& +2L_{1234}H_{12}H_{35}H_{46}\frac{\delta\Gamma_I}{\delta H_{56}} - \frac{2}{3}L_{1234}H_{15}H_{26}\frac{\delta\Gamma_I}{\delta H_{56}}H_{37}H_{48}\frac{\delta\Gamma_I}{\delta H_{78}} \\
& -\frac{2}{3}L_{1234}H_{15}H_{26}H_{37}H_{48}\Gamma^{HH}_{5678} + 2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{73}\pi_4 \\
& +2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\Gamma^G_{89}G_{93}\pi_4 \\
& -2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta\pi_9}G_{93}\pi_4 \\
& +4M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar{1}}}G_{\bar{1}3}\pi_4 \\
& +2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar{1}}}G_{\bar{1}3}\pi_4 \\
& -4M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta\Gamma_I}{\delta H_{0\bar{1}}}H_{\bar{1}\bar{2}}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar{2}}\delta\pi_{\bar{3}}}G_{\bar{3}3}\pi_4 \\
& +2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\Gamma^G_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta\sigma_{\bar{1}}}H_{\bar{1}\bar{2}}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar{2}}\delta\pi_{\bar{3}}}G_{\bar{3}3}\pi_4 \\
& -4M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\Gamma^G_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta\sigma_{\bar{1}}}H_{\bar{1}\bar{2}}\frac{\delta\Gamma_I}{\delta H_{\bar{2}\bar{3}}}H_{\bar{3}\bar{4}}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar{4}}\delta\pi_{\bar{5}}}G_{\bar{5}3}\pi_4 \\
& -2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta\pi_9}G_{90}\Gamma^G_{0\bar{1}}G_{\bar{1}3}\pi_4 \\
& +4M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar{1}}}G_{\bar{1}\bar{2}}\Gamma^G_{\bar{2}\bar{3}}G_{\bar{3}3}\pi_4 \\
& +2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar{1}}}G_{\bar{1}\bar{2}}\Gamma^G_{\bar{2}\bar{3}}G_{\bar{3}3}\pi_4 \\
& -4M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta\Gamma_I}{\delta H_{0\bar{1}}}H_{\bar{1}\bar{2}}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar{2}}\delta\pi_{\bar{3}}}G_{\bar{3}\bar{4}}\Gamma^G_{\bar{4}\bar{5}}G_{\bar{5}3}\pi_4
\end{aligned}
$$

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$$
\begin{aligned}
&+2M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\Gamma^G_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta\sigma_{\bar1}}H_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar2}\delta\pi_{\bar3}}G_{\bar3\bar4}\Gamma^G_{\bar4\bar5}G_{\bar5 3}\pi_4\\
&-4M_{1234}H_{15}H_{26}\frac{\delta^2\Gamma_I}{\delta H_{56}\delta\pi_7}G_{78}\Gamma^G_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta\sigma_{\bar1}}H_{\bar1\bar2}\frac{\delta\Gamma_I}{\delta H_{\bar2\bar3}}H_{\bar3\bar4}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar4}\delta\pi_{\bar5}}\\
&\qquad\times G_{\bar5\bar6}\Gamma^G_{\bar6\bar7}G_{\bar7 3}\pi_4\\
&+M_{1234}H_{15}H_{26}G_{37}G_{48}\Gamma^{HG}_{5678},
\end{aligned}
\tag{20}
$$

where it is understood that in all terms we have set $\sigma = 0$.

For diagrams without Goldstone field π or propagator G, we may also use the identities in Ref. [5], of course.

We still have to define the quantities $\Gamma^{\sigma H}$, $\Gamma^{\sigma G}$, Γ^{HH}, Γ^{HG}, and Γ^G appearing in the identities (18)-(20). The first four are given by

$$
\begin{aligned}
\Gamma^{\sigma H}_{123} = {}& \frac{\delta^2\Gamma_I}{\delta\sigma_1\delta H_{23}} - \frac{\delta^2\Gamma_I}{\delta\sigma_1\delta\pi_4}G_{45}\frac{\delta^2\Gamma_I}{\delta\pi_5\delta H_{23}} - \frac{\delta^2\Gamma_I}{\delta\sigma_1\delta\pi_4}G_{45}\Gamma^G_{56}G_{67}\frac{\delta^2\Gamma_I}{\delta\pi_7\delta H_{23}}\\
&-2\frac{\delta\Gamma_I}{\delta H_{14}}H_{45}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta H_{23}} + 2\frac{\delta\Gamma_I}{\delta H_{14}}H_{45}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta\pi_6}G_{67}\frac{\delta^2\Gamma_I}{\delta\pi_7\delta H_{23}}\\
&+2\frac{\delta\Gamma_I}{\delta H_{14}}H_{45}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta\pi_6}G_{67}\Gamma^G_{78}G_{89}\frac{\delta^2\Gamma_I}{\delta\pi_9\delta H_{23}},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\Gamma^{\sigma G}_{123} = {}& \frac{\delta\Gamma_I}{\delta\sigma_1\delta G_{23}} - \frac{\delta^2\Gamma_I}{\delta\sigma_1\delta\pi_4}G_{45}\frac{\delta^2\Gamma_I}{\delta\pi_5\delta G_{23}} - \frac{\delta^2\Gamma_I}{\delta\sigma_1\delta\pi_4}G_{45}\Gamma^G_{56}G_{67}\frac{\delta^2\Gamma_I}{\delta\pi_7\delta G_{23}}\\
&-2\frac{\delta\Gamma_I}{\delta H_{14}}H_{45}\frac{\delta\Gamma_I}{\delta\sigma_5\delta G_{23}} + 2\frac{\delta\Gamma_I}{\delta H_{14}}H_{45}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta\pi_6}G_{67}\frac{\delta^2\Gamma_I}{\delta\pi_7\delta G_{23}}\\
&+2\frac{\delta\Gamma_I}{\delta H_{14}}H_{45}\frac{\delta^2\Gamma_I}{\delta\sigma_5\delta\pi_6}G_{67}\Gamma^G_{78}G_{89}\frac{\delta^2\Gamma_I}{\delta\pi_9\delta G_{23}},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\Gamma^{HH}_{1234} = {}& -\frac{\delta^2\Gamma_I}{\delta H_{12}\delta H_{34}} + \frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta H_{34}}\\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta H_{34}} + \frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta H_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta\Gamma_I}{\delta H_{67}}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta H_{34}}\\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta\pi_9}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta H_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar1}}G_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar2}\delta H_{34}}
\end{aligned}
$$

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$$
\begin{aligned}
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta\Gamma_I}{\delta H_{0\bar1}}H_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar2}\delta\pi_{\bar3}}G_{\bar3\bar4}\Gamma^G_{\bar4\bar5}G_{\bar5\bar6}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar6}\delta H_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta H_{34}}\\
&+4\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta\Gamma_I}{\delta H_{67}}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta H_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta\pi_7}G_{78}\Gamma^G_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta H_{34}}\\
&+4\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta\Gamma_I}{\delta H_{67}}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta\pi_9}G_{90}\Gamma^G_{0\bar1}G_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar2}\delta H_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta\sigma_9}G_{90}\Gamma^G_{0\bar1}G_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar2}\delta H_{34}}\\
&+4\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\sigma_{\bar1}}G_{\bar1\bar2}\Gamma^G_{\bar2\bar3}G_{\bar3\bar4}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar4}\delta H_{12}}, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{HG}_{1234} = &-\frac{\delta^2\Gamma_I}{\delta H_{12}\delta G_{34}}+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta G_{34}}\\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma}{\delta\pi_8\delta G_{34}}-\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta G_{34}}\\
&+2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta\Gamma_I}{\delta H_{67}}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta G_{34}}+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta G_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta G_{34}}\\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\sigma_9}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta G_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\sigma_9}H_{90}\frac{\delta\Gamma_I}{\delta H_{0\bar1}}H_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar2}\delta G_{34}}\\
&-\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\pi_7}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta G_{34}}\\
&+2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta\Gamma_I}{\delta H_{67}}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\pi_9}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta G_{34}}\\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\pi_9}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta G_{34}}\\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\pi_{\bar1}}G_{\bar1\bar2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar2}\delta G_{34}}
\end{aligned}
$$

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(World Scientific, Singapore, 2001).

$$
\begin{aligned}
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar 1}}G_{\bar 1\bar 2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 2}\delta G_{34}} \\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta\Gamma_I}{\delta H_{0\bar 1}}H_{\bar 1\bar 2}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar 2}\delta\pi_{\bar 3}}G_{\bar 3\bar 4}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 4}\delta G_{34}} \\
&-\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta^2\Gamma_I}{\delta\sigma_6\delta\pi_7}G_{78}\Gamma^G_{89}G_{90}\frac{\delta^2\Gamma_I}{\delta\pi_0\delta G_{34}} \\
&+2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\sigma_5}H_{56}\frac{\delta\Gamma_I}{\delta H_{67}}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\delta\pi_9}G_{90}\Gamma^G_{0\bar 1}G_{\bar 1\bar 2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 2}\delta G_{34}} \\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta^2\Gamma_I}{\delta\sigma_8\pi_9}G_{90}\Gamma^G_{0\bar 1}G_{\bar 1\bar 2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 2}\delta G_{34}} \\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\frac{\delta^2\Gamma_I}{\delta\pi_6\delta\sigma_7}H_{78}\frac{\delta\Gamma_I}{\delta H_{89}}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\pi_{\bar 1}}G_{\bar 1\bar 2}\Gamma^G_{\bar 2\bar 3}G_{\bar 3\bar 4}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 4}\delta G_{34}} \\
&+\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta^2\Gamma_I}{\delta\sigma_0\delta\pi_{\bar 1}}G_{\bar 1\bar 2}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 2}\delta G_{34}} \\
&-2\frac{\delta^2\Gamma_I}{\delta H_{12}\delta\pi_5}G_{56}\Gamma^G_{67}G_{78}\frac{\delta^2\Gamma_I}{\delta\pi_8\delta\sigma_9}H_{90}\frac{\delta\Gamma_I}{\delta H_{0\bar 1}}H_{\bar 1\bar 2}\frac{\delta^2\Gamma_I}{\delta\sigma_{\bar 2}\delta\pi_{\bar 3}}G_{\bar 3\bar 4}\frac{\delta^2\Gamma_I}{\delta\pi_{\bar 4}\delta G_{34}},
\end{aligned}
\tag{24}
$$

while for Γ^G holds

$$
\Gamma^G_{12} = -\frac{\delta^2\Gamma_I}{\delta\pi_1\delta\pi_2} - \Gamma^G_{13}G_{34}\frac{\delta^2\Gamma_I}{\delta\pi_4\delta\pi_2}. \tag{25}
$$

To obtain recursion relations for the 1PI diagrams, it is most useful to separate Eqs. (18)-(25) by numbers of loops and by powers of σ and π. The resulting equations recursively generate all 1PI Feynman graphs, including the vacuum graphs (i.e. graphs with zero powers of σ and π).

We can hardly praise our results for conciseness. However, once the derivation of our equations as well as their implementation in terms of recursion relations are automated, they provide a straightforward and safe means of obtaining all 1PI Feynman graphs for the spontaneously broken $O(N)$ ϕ^4-theory. Nevertheless it would be very useful if one could find a simpler set of equations, a task left for future work.

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