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VASSILIEV INVARIANTS AND FUNCTIONAL INTEGRATION

L.H. KAUFFMAN

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045, USA

E-mail: kauffman@uic.edu

This paper, dedicated to Hagen Kleinert, is an exposition of the relationship between the Witten functional integral and the theory of Vassiliev invariants of knots and links in three-dimensional space.

1 Introduction

In this article we want to show how Vassiliev invariants in knot theory arise naturally in context of the Witten functional integral. The relationship between Vassiliev invariants and the Witten integral has been known since Bar-Natan's thesis [1] where he discovered, through this connection, how to define Lie algebraic weight systems for these invariants.

The paper is a sequel to the Refs. [2–4] and an expanded version of a talk given at The Fifth Taiwan International Symposium on Statistical Physics (August 1999). In these papers we show more about the relationship of Vassiliev invariants and the Witten functional integral. In particular, we investigate how the Kontsevich integrals, used to give rigorous definitions of these invariants, arise as Feynman integrals in the perturbative expansion of the Witten functional integral; see also the work of Labastida and Pérez [5] on this same subject. Their work comes to an identical conclusion, interpreting the Kontsevich integrals in terms of the light-cone gauge and thereby extending the original work of Fröhlich and King [6]. The purpose of this paper is to give an exposition of the beginnings of these relationships and to introduce diagrammatic techniques that illuminate the connections.

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This article is divided into two sections. First we discuss Vassiliev invariants and invariants of rigid vertex graphs and then we introduce the basic formalism and show how the functional integral is related directly to Vassiliev invariants.

It gives me great pleasure to dedicate this paper to Hagen Kleinert. His pioneering work [7] in the applications of functional integration to physical problems and his interest in topological work has sustained my own interest in this field since we met in 1996 in a conference on this very topic held in Cargese, Corsica, and organized by Pierre Cartier and Cécile DeWitt-Morette [8].

2 Vassiliev Invariants and Invariants of Rigid Vertex Graphs

If $V(K)$ is a Laurent polynomial valued, or more generally, commutative ring valued invariant of knots, then it can be naturally extended to an invariant of rigid vertex graphs [9] by defining the invariant of graphs in terms of the knot invariant via an "unfolding" of the vertex. That is, we can regard the vertex as a "black box" and replace it by any tangle of our choice. Rigid vertex motions of the graph preserve the contents of the black box, and hence implicate ambient isotopies of the link obtained by replacing the black box by its contents. Invariants of knots and links that are evaluated on these replacements are then automatically rigid vertex invariants of the corresponding graphs. If we set up a collection of multiple replacements at the vertices with standard conventions for the insertions of the tangles, then a summation over all possible replacements can lead to a graph invariant with new coefficients corresponding to the different replacements. In this way each invariant of knots and links implicates a large collection of graph invariants [9,10]. The simplest tangle replacements for a 4-valent vertex are the two crossings, positive and negative, and the oriented smoothing. Let $V(K)$ be any invariant of knots and links. Extend V to the category of rigid vertex embeddings of 4-valent graphs by the formula

$$V(K_*) = aV(K_+) + bV(K_-) + cV(K_0)\,, \tag{1}$$

where K_+ denotes a knot diagram K with a specific choice of positive crossing, K_- denotes a diagram identical to the first with the positive crossing replaced by a negative crossing, and K_* denotes a diagram identical to the first with the positive crossing replaced by a graphical node. This formula means that

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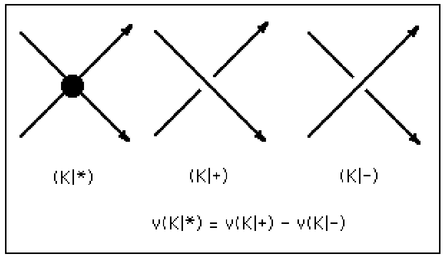

Figure 1. Exchange identity for Vassiliev invariants.

we define $V(G)$ for an embedded 4-valent graph G by taking the sum

$$V(G) = \sum_S a^{i_+(S)} b^{i_-(S)} c^{i_0(S)} V(S)\,, \tag{2}$$

with the summation over all knots and links S obtained from G by replacing a node of G with either a crossing of positive or negative type, or with a smoothing of the crossing that replaces it by a planar embedding of non-touching segments (denoted 0). It is not hard to see that if $V(K)$ is an ambient isotopy invariant of knots, then, this extension is a rigid vertex isotopy invariant of graphs. In rigid vertex isotopy the cyclic order at the vertex is preserved, so that the vertex behaves like a rigid disk with flexible strings attached to it at specific points.

There is a rich class of graph invariants that can be studied in this manner. The Vassiliev invariants [11,12] constitute the important special case of these graph invariants where $a = +1$, $b = -1$ and $c = 0$:

$$V(K_*) = V(K_+) - V(K_-). \tag{3}$$

Call this formula the *exchange identity* for the Vassiliev invariant V (see Fig. 1). Thus $V(G)$ is a Vassiliev invariant if V is said to be of *finite type k* if $V(G) = 0$ whenever $|G| > k$ where $|G|$ denotes the number of (4-valent) nodes in the graph G. The notion of finite type is of extraordinary significance in studying these invariants. One reason for this is the following basic Lemma.

Lemma. If a graph G has exactly k nodes, then the value of a Vassiliev invariant v_k of type k on G, i.e. $v_k(G)$, is independent of the embedding of G.

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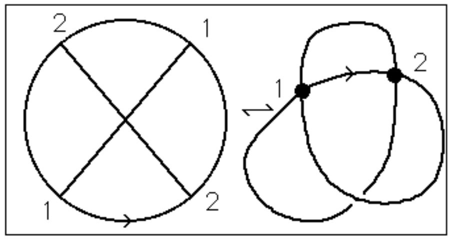

Figure 2. Chord diagrams.

Proof. The different embeddings of G can be represented by link diagrams with some of the 4-valent vertices in the diagram corresponding to the nodes of G. It suffices to show that the value of $v_k(G)$ is unchanged under switching of a crossing. However, the exchange identity for v_k shows that this difference is equal to the evaluation of v_k on a graph with $k+1$ nodes and hence is equal to zero. This completes the proof.

The upshot of this Lemma is that Vassiliev invariants of type k are intimately involved with certain abstract evaluations of graphs with k nodes. In fact, there are restrictions (the four-term relations) on these evaluations demanded by the topology and it follows from results of Kontsevich [12] that such abstract evaluations actually determine the invariants. The knot invariants derived from classical Lie algebras are all built from Vassiliev invariants of finite type. All of this is directly related to the Witten functional integral [13].

In the next few figures we illustrate some of these main points. In Fig. 2 we show how one associates a so-called chord diagram to represent the abstract graph associated with an embedded graph. The chord diagram is a circle with arcs connecting those points on the circle that are welded to form the corresponding graph. In Fig. 3 we illustrate how the four-term relation is a consequence of topological invariance. In Fig. 4 we show how the four-term relation is a consequence of the abstract pattern of the commutator identity for a matrix Lie algebra. This shows that the four-term relation is directly related to a categorical generalisation of Lie algebras. Figure 5 illustrates how the weights are assigned to the chord diagrams in the Lie algebra case by inserting Lie algebra matrices into the circle and taking a trace of a sum of matrix products.

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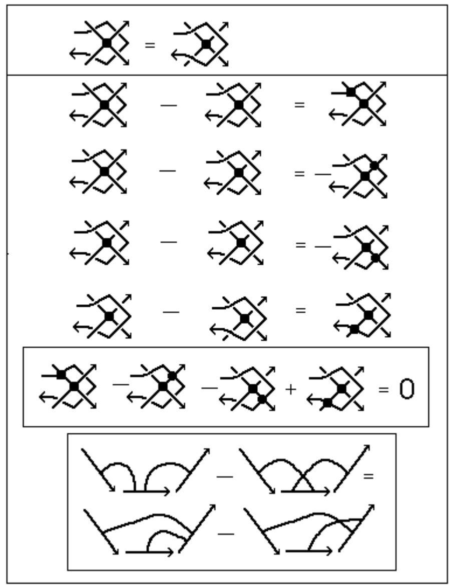

Figure 3. The four-term relation from topology.

3 Vassiliev Invariants and Witten Functional Integral

In Ref. [13] Edward Witten proposed a formulation of a class of 3-manifold invariants (see also Ref. [14]) which generalize Feynman integrals taking the form

$$Z(M) = \int DAe^{(ik/4\pi)S(M,A)} \,. \tag{4}$$

Here M denotes a 3-manifold without boundary and A is a gauge field, also called a gauge potential or gauge connection, defined on M. The gauge field is a one-form on a trivial G-bundle over M with values in a representation of the Lie algebra of G. The group G corresponding to this Lie algebra is said to be the gauge group. In this integral the action $S(M, A)$ is taken to be the integral over M of the trace of the Chern-Simons three-form $A \wedge dA + (2/3)A \wedge A \wedge A$, where the product is the wedge product of differential forms. $Z(M)$ integrates over all gauge fields modulo gauge equivalence. The formalism and internal logic of Witten's integral supports the existence of a large class of topological invariants of 3-manifolds and associated invariants of knots and links in these

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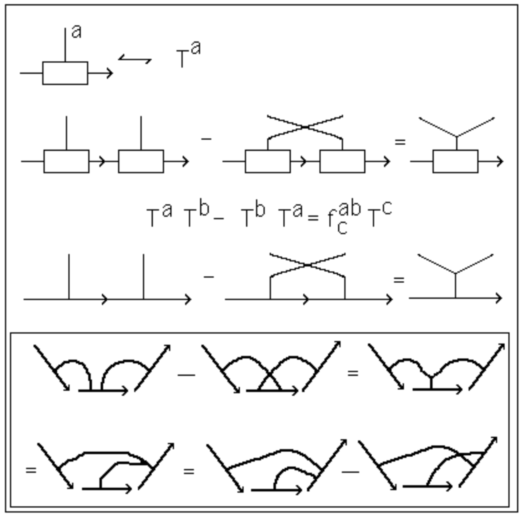

Figure 4. The four-term relation from categorical Lie algebra.

manifolds.

The invariants associated with this integral have been given rigorous combinatorial descriptions but questions and conjectures arising from the integral formulation are still outstanding. Specific conjectures about this integral take the form of just how it implicates invariants of links and 3-manifolds, and how these invariants behave in certain limits of the coupling constant k in the integral. Many conjectures of this sort can be verified through the combinatorial models. On the other hand, the really outstanding conjecture about the integral is that it exists! At the present time there is no measure theory or generalization of measure theory that supports it. Here is a formal structure of great beauty. It is also a structure whose consequences can be verified by a remarkable variety of alternative means.

We now look at the formalism of the Witten functional integral in more detail and see how it implicates invariants of knots and links corresponding to each classical Lie algebra. In order to accomplish this task, we need to introduce the Wilson loop. The Wilson loop is an exponentiated version of integrating the gauge field along a loop K in three-space that we take to be an embedding (knot) or a curve with transversal self-intersections. For this discussion, the Wilson loop will be denoted by the notation $W_K(A) = \langle K|A\rangle$

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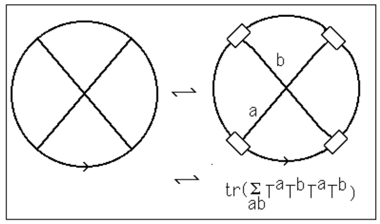

Figure 5. Calculating Lie algebra weights.

to stress the dependence on the loop K and the field A. It is usually indicated by the symbolism

$$W_K(A) = \langle K|A\rangle = \mathrm{tr}\left(Pe^{\oint_K A}\right) . \tag{5}$$

Here P denotes path ordered integration. As we are integrating and exponentiating matrix valued functions, we must keep track of the order of the operations. The symbol tr denotes the trace of the resulting matrix. With the help of the Wilson loop functional on knots and links, Witten writes down a functional integral for link invariants in a 3-manifold M:

$$Z(M,K) = \int DAe^{(ik/4\pi)S(M,A)}\mathrm{tr}\left(Pe^{\oint_K A}\right) = \int DAe^{(ik/4\pi)S}\langle K|A\rangle . \tag{6}$$

Here $S(M,A)$ is the Chern-Simons Lagrangian as in the previous discussion. We abbreviate $S(M,A)$ as S and write $\langle K|A\rangle$ for the Wilson loop. Unless otherwise mentioned, the manifold M will be the three-dimensional sphere S^3.

An analysis of the formalism of this functional integral reveals quite a bit about its role in knot theory. This analysis depends upon key facts relating the curvature of the gauge field to both the Wilson loop and the Chern-Simons Lagrangian. The idea for using the curvature in this way is due to Lee Smolin [15] (see also Ref. [16]). To this end, let us recall the local coordinate structure of the gauge field $A(x)$, where x is a point in three-space. We can write $A(x) = A^a_k(x)T_a dx^k$ where the index a ranges from 1 to m with the Lie algebra basis $\{T_1, T_2, T_3, ..., T_m\}$ and the index k goes from 1 to 3. For each choice of a and k, $A^a_k(x)$ is a smooth function defined on three-space. In $A(x)$ we sum over the values of repeated indices. The Lie algebra generators T_a are matrices corresponding to a given representation of the Lie algebra of the gauge group G. We assume some properties of these matrices as follows:

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(1) $[T_a, T_b] = if^{abc}T_c$ where $[x, y] = xy - yx$, and the matrix of structure constants f^{abc} is totally antisymmetric. There is summation over repeated indices.
(2) $\mathrm{tr}(T_aT_b) = \delta_{ab}/2$ where δ_{ab} is the Kronecker delta, i.e. $\delta_{ab} = 1$ if $a = b$ and zero otherwise.

We also assume some facts about curvature (the reader may enjoy comparing with the exposition in Ref. [17]; but note the difference of conventions on the use of i in the Wilson loops and the curvature definitions). The first fact is the relation of Wilson loops and curvature for small loops:

Fact 1. The result of evaluating a Wilson loop about a very small planar circle around a point x is proportional to the area enclosed by this circle times the corresponding value of the curvature tensor of the gauge field evaluated at x. The curvature tensor is written $F^a_{rs}(x)T_a dx^r dy^s$. It is the local coordinate expression of $F = dA + A \wedge A$.
Application of Fact 1. Consider a given Wilson line $\langle K|S\rangle$. Ask how its value will change if it is deformed infinitesimally in the neighborhood of a point x on the line. Approximate the change according to Fact 1, and regard the point x as the place of the curvature evaluation. Let $\delta\langle K|A\rangle$ denote the change in the value of the line. $\delta\langle K|A\rangle$ is given by the formula

$$\delta\langle K|A\rangle = dx^r dx^s F_a^{rs}(x) T_a \langle K|A\rangle. \tag{7}$$

This is the first-order approximation to the change in the Wilson line. In this formula it is understood that the Lie algebra matrices T_a are to be inserted into the Wilson line at the point x, and that we are summing over repeated indices. This means that each $T_a\langle K|A\rangle$ is a new Wilson line obtained from the original line $\langle K|A\rangle$ by leaving the form of the loop unchanged, but inserting the matrix T_a into that loop at the point x. In Fig. 6 we have illustrated this mode of insertion of Lie algebra into the Wilson loop. Here and in further illustrations in this section we use $W_K(A)$ to denote the Wilson loop. Note that in the diagrammatic version shown in Fig. 6 we have let small triangles with legs indicate dx^i. The legs correspond to indices just as in our work in the last section with Lie algebras and chord diagrams. The curvature tensor is indicated as a circle with three legs corresponding to the indices of F_a^{rs}.

Notation. In the diagrams in this section we have dropped mention-

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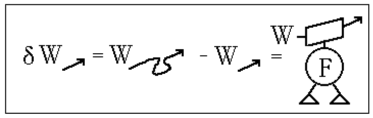

Figure 6. Lie algebra and curvature tensor insertion into the Wilson loop.

ing the factor of $1/4\pi$ that occurs in the integral. This convention saves space in the figures. In these figures L denotes the Chern-Simons Lagrangian.

Remark. In thinking about the Wilson line (5), it is helpful to recall Euler's formula for the exponential:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n . \tag{8}$$

The Wilson line is the limit, over partitions of the loop K, of products of the matrices $(1 + A(x))$ where x runs over the partition. Thus we can write symbolically

$$\langle K|A\rangle = \prod_{x \in K} (1 + A(x)) = \prod_{x \in K} (1 + A_k^a(x) T_a dx^k) . \tag{9}$$

It is understood that a product of matrices around a closed loop connotes the trace of the product. The ordering is forced by the one-dimensional nature of the loop. Inserting a given matrix into this product at a point on the loop is then a well-defined concept. If T is a given matrix then it is understood that $T\langle K|A\rangle$ denotes the insertion of T into some point of the loop. In the case above, it is understood from the context in the formula that the insertion is to be performed at the point x indicated in the argument of the curvature.

Remark. The previous remark implies the following formula for the variation of the Wilson loop with respect to the gauge field:

$$\frac{\delta \langle K|A\rangle}{\delta(A_k^a(x))} = dx^k T_a \langle K|A\rangle . \tag{10}$$

Varying the Wilson loop with respect to the gauge field results in inserting an infinitesimal Lie algebra element into the loop. Figure 7 gives a diagrammatic form for this formula. In that figure we use a capital D with up and down legs to denote the derivative $\delta/\delta(A_k^a(x))$. Insertions in the Wilson line are

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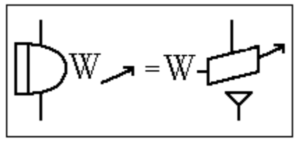

Figure 7. Differentiating the Wilson line.

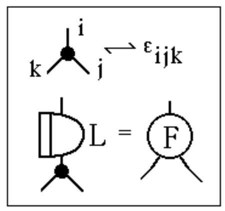

Figure 8. Variational formula for curvature.

indicated directly by matrix boxes placed in a representative bit of line.

PROOF.

$$\frac{\delta\langle K|A\rangle}{\delta(A_k^a(x))} = \frac{\delta}{\delta(A_k^a(x))}\prod_{y\in K}(1+A_k^a(y)T_a dy^k) = \prod_{y<x\in K}(1+A_k^a(y)T_a dy^k)T_a dx^k$$
$$\times \prod_{y>x\in K}(1+A_k^a(y)T_a dy^k) = dx^k T_a\langle K|A\rangle. \qquad (11)$$

FACT 2. The variation of the Chern-Simons Lagrangian S with respect to the gauge potential at a given point in three-space is related to the values of the curvature tensor at that point by the following formula:

$$F_{rs}^a(x) = \epsilon_{rst}\frac{\delta S}{\delta(A_t^a(x))}\,. \qquad (12)$$

Here ϵ_{abc} is the epsilon symbol for three indices, i.e. it is $+1$ for positive permutations of 123 and -1 for negative permutations of 123 and zero if any two indices are repeated. A diagrammatic representation for this formula is shown in Fig. 8. With these facts at hand we are prepared to determine how the Witten functional integral behaves under a small deformation of the loop K.

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THEOREM 1. Let $Z(K) = Z(S^3, K)$ and let $\delta Z(K)$ denote the change of $Z(K)$ under an infinitesimal change in the loop K. Then

$$\delta Z(K) = (4\pi i/k)\int dA e^{(ik/4\pi)S}[\mathrm{Vol}]T_aT_a\langle K|A\rangle\,, \tag{13}$$

where $\mathrm{Vol} = \epsilon_{rst}dx^r dx^s dx^t$. The sum is taken over repeated indices, and the insertion is taken of the matrices T_aT_a at the chosen point x on the loop K that is regarded as the center of the deformation. The volume element $\mathrm{Vol} = \epsilon_{rst}dx^r dx^s dx^t$ is taken with regard to the infinitesimal directions of the loop deformation from this point on the original loop.

THEOREM 2. The same formula applies, with a different interpretation, to the case, where x is a double point of transversal self-intersection of a loop K, and the deformation consists in shifting one of the crossing segments perpendicularly to the plane of intersection so that the self-intersection point disappears. In this case, one T_a is inserted into each of the transversal crossing segments so that $T_aT_a\langle K|A\rangle$ denotes a Wilson loop with a self-intersection at x and insertions of T_a at $x+\epsilon_1$ and $x+\epsilon_2$ where ϵ_1 and ϵ_2 denote small displacements along the two arcs of K that intersect at x. In this case, the volume form is nonzero, with two directions coming from the plane of movement of one arc, and the perpendicular direction is the direction of the other arc.

PROOF.

$$\begin{aligned}
\delta Z(K) &= \int DA e^{(ik/4\pi)S}\delta\langle K|A\rangle = \int DA e^{(ik/4\pi)S} dx^r dy^s F^a_{rs}(x) T_a\langle K|A\rangle \\
&= \int DA e^{(ik/4\pi)S} dx^r dy^s \epsilon_{rst}\frac{\delta S}{\delta(A^a_t(x))}T_a\langle K|A\rangle \\
&= (-4\pi i/k)\int DA \frac{\delta e^{(ik/4\pi)S}}{\delta(A^a_t(x))}\epsilon_{rst}dx^r dy^s T_a\langle K|A\rangle \\
&= (4\pi i/k)\int DA e^{(ik/4\pi)S}\epsilon_{rst}dx^r dy^s\frac{\delta T_a\langle K|A\rangle}{\delta(A^a_t(x))} \\
&= (4\pi i/k)\int DA e^{(ik/4\pi)S}[\mathrm{Vol}]T_aT_a\langle K|A\rangle\,.
\end{aligned} \tag{14}$$

This completes the formalism of the proof. In the case of part 2, a change of interpretation occurs at the point in the argument when the Wilson line is differentiated. Differentiating a self-intersecting Wilson line at a point of self-intersection is equivalent to differentiating the corresponding product of

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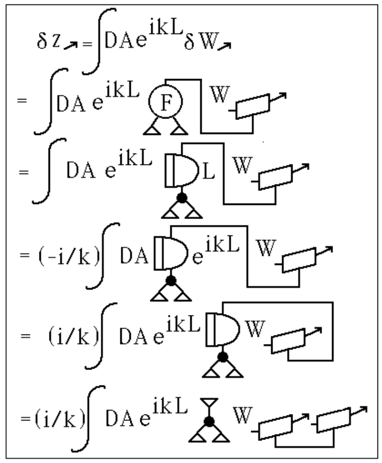

Figure 9. Varying the functional integral by varying the line.

matrices with respect to a variable that occurs at two points in the product (corresponding to the two places where the loop passes through the point). One of these derivatives gives rise to a term with volume form equal to zero, the other term is the one that is described in part 2. This completes the proof of the theorem.

The formalism of this proof is illustrated in Fig. 9. In the case of switching a crossing the key point is to write the crossing-switch as a composition of first moving a segment to obtain a transversal intersection of the diagram with itself, and then to continue the motion to complete the switch. One then analyzes separately the case where x is a double point of transversal self-intersection of a loop K, and the deformation consists in shifting one of the crossing segments perpendicularly to the plane of intersection so that the self-intersection point disappears. In this case, one T_a is inserted into each of the transversal crossing segments so that $T^a T^a \langle K|A\rangle$ denotes a Wilson loop with a self-intersection at x and insertions of T^a at $x + \epsilon_1$ and $x + \epsilon_2$ as in part 2 of the theorem above. The first insertion is in the moving line, due to curvature. The second insertion is the consequence of differentiating the self-touching Wilson line. Since this line can be regarded as a product, the differentiation occurs twice at the point of intersection, and it is the second direction that produces the non-vanishing volume form.

Up to the choice of our conventions for constants, the switching formula,

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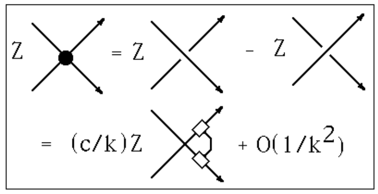

Figure 10. The difference formula.

as shown in Fig. 10, reads

$$\begin{aligned} Z(K_+) - Z(K_-) &= (4\pi i/k) \int DA e^{(ik/4\pi)S} T_a T_a \langle K_{**}|A\rangle \\ &= (4\pi i/k) Z(T^a T^a K_{**})\,, \end{aligned} \tag{15}$$

where K_{**} denotes the result of replacing the crossing by a self-touching crossing. We distinguish this from adding a graphical node at this crossing by using the double star notation. A key point is to notice that the Lie algebra insertion for this difference is exactly what is done (in chord diagrams) to make the weight systems for Vassiliev invariants (without the framing compensation). Here we take formally the perturbative expansion of the Witten functional integral to obtain Vassiliev invariants as coefficients of the powers of $1/k^n$. Thus the formalism of the Witten functional integral takes one directly to these weight systems in the case of the classical Lie algebras. In this way the functional integral is central to the structure of the Vassiliev invariants.

Acknowledgments

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