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PATH INTEGRALS IN QUANTUM COSMOLOGY

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Path integrals in quantum cosmology differ both in formulation and interpretation from ordinary path integrals. After reviewing the highlights of these differences, I shall discuss in detail a particular model with a cosmological constant. This example contains some interesting features connected with space-time transformations in the path integral.

1 Introduction

Path integrals play a prominent role in modern physics [1]. Their application ranges from quantum mechanics, statistical mechanics up to quantum field theory. It thus seems natural to apply path integration also to quantum gravity, a theory being still in the process of construction.

In this context, interest focuses mainly on two issues. First, path integrals are applied to the full gravitational field in order to find a nonperturbative formulation of quantum gravity. This is also a convenient starting point for lattice formulations. Second, path integrals are frequently formulated for finite-dimensional models of quantum cosmology in order to study features of the early Universe, in particular in connection with an inflationary phase. Here, applications are usually made for an energy regime somewhat lower than the Planck scale. This gives rise to the hope that the results are independent of the unknown behavior of the full theory.

This contribution to Hagen Kleinert's *Festschrift* focuses on the second application. In the next section I will review the main properties of a quantum cosmological path integral and discuss some results for a simple model – the indefinite harmonic oscillator. Interesting consequences are drawn for the

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meaning of the *no-boundary condition* in quantum cosmology. The third section contains a model that has not yet been studied in the present context – a Friedmann model with a cosmological constant. In the evaluation of the path integral, it is necessary to take into account the curved nature of the configuration space. Some technical details are relegated to the Appendix.

2 General Properties of Quantum Cosmological Path Integrals

In quantum cosmology one focuses on finite-dimensional models whose Hamiltonian is given by

$$H = \frac{1}{2}G^{ab}(q)p_a p_b + V(q) \ , \tag{1}$$

where q is a shorthand notation for n degrees of freedom. These can be the scale factor(s) of a cosmological model plus homogeneous matter variables. Classically, this Hamiltonian is constrained to vanish as a consequence of the reparametrization invariance of general relativity. The coefficients G^{ab} denote the (inverse) metric on configuration space (DeWitt metric or superspace metric). For a discussion of its features see e.g. Ref. [2]. In the canonical version of quantum gravity, the classical constraint is turned into a condition on allowed wave functions, the so-called Wheeler-DeWitt equation

$$\hat{H}\psi(q) = 0 \ . \tag{2}$$

The corresponding path integral has to be formulated within the theory of constrained systems, i.e. including gauge fixing and Faddeev-Popov ghosts. In the present finite-dimensional case, the procedure can be highly simplified and leads to the following path integral (for a review see e.g. Ref. [3])

$$\begin{aligned} G(q',q'') &= \int dT \int Dp_a Dq^a \ \exp\left(i\int_0^T dt \ (p_a\dot{q}^a - H)\right) \\ &= \int dT \ \langle q'',T|q',0\rangle \ , \end{aligned} \tag{3}$$

where $\langle q'',T|q',0\rangle$ is an ordinary quantum mechanical propagator which satisfies the Schrödinger equation. Since quantum gravity does not contain any external time parameter, time must be "integrated out" to get the correct path integral. The range of the T-integration in Eq. (3) is not fixed a priori. Since integration along the real axis leads in general to a divergent result, the idea is to look for contours in the *complex* T-plane that render the integral

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convergent [3,4]. In order to prevent a violation of gauge invariance, attention is restricted to either infinite contours, half-infinite contours starting at zero, or closed loops around the origin [3]. Infinite contours and closed loops should then lead to *solutions* of Eq. (2), while half-infinite contours should yield *Green functions*.

Due to the T-integration in Eq. (3), path integrals in quantum cosmology have different properties than quantum mechanical path integrals [5]. In fact, they behave more like *energy Green functions* [1] rather than propagators, as was first emphasised by Hájíček [6]. For this reason, they do not obey any composition law, which means that they are not intimately connected with an external time variable. A simpler but similar analogy for the path integral in Eq. (3) is provided by the proper-time representation for the relativistic particle [5]. Due to these properties of the path integral, the central interest focused on the relation between *boundary conditions* and the choice of metrics to be integrated over [7]. The famous "no-boundary proposal" by Hartle and Hawking [8] aimed at finding a *unique* solution to the Wheeler-DeWitt equation (2) by integrating only over *compact* ("finite") geometries. This should have yielded the "wave function of the Universe" (for a critical assessment concerning physical interpretations of the no-boundary wave function see, e.g. Ref. [10]), but it did not. Even when integrating over complex metrics corresponding to the complex T-integration above, one gets a whole *class* of solutions to Eq. (2).

Still, it is of interest to investigate this distinguished class of solutions in more detail. The question posed in Refs. [5,9], for example, was: Can one get special cases which reflect the properties of classical solutions from the class of no-boundary wave functions, i.e. can one construct wave packets that follow a classical trajectory in configuration space?

The simplest nontrivial model to investigate this question is the *indefinite oscillator* which has the advantage of being exactly soluble. The Wheeler-DeWitt equation (2) then reads

$$\hat{H}\psi \equiv \frac{1}{2}\left(\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial \chi^2} - a^2 + \chi^2\right)\psi(a,\chi) = 0 \ , \tag{4}$$

where a denotes the scale factor of a closed Friedmann Universe, and χ a (rescaled) conformally coupled field. The classical solutions of this model would consist of trajectories ("Lissajous ellipses") that are confined to a rectangle around the origin. The interesting point is that the path integral (3) can be exactly evaluated for this model. The path integration over Da and

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$D\chi$ gives the usual result for the harmonic oscillator [1], but with the relative sign between the two oscillators reversed. The remaining T-integration then reads

$$G(a'', \chi''; a', \chi') = \frac{1}{2\pi} \int \frac{dT}{\sin T} \exp\left(\frac{i}{2\sin T}(Q_1 \cos T + Q_2) \right) , \tag{5}$$

with

$$Q_1 \equiv \chi''^2 + \chi'^2 - (a''^2 + a'^2), \tag{6}$$

$$Q_2 \equiv -2\chi'\chi'' + 2a'a'' . \tag{7}$$

One can interpret the no-boundary proposal as imposing the values $a' = \chi' = 0$ and looking for contours in the T-plane that render the integral convergent. It turns out [5] that for half-infinite contours the result of Eq. (5) is

$$G_1(a, \chi) = \frac{1}{2\pi} K_0 \left(\frac{|a^2 - \chi^2|}{2} \right) , \tag{8}$$

where K_0 denotes the modified Bessel function, while for an infinite contour the result is

$$G_2(a, \chi) = \frac{1}{2} I_0 \left(\frac{a^2 - \chi^2}{2} \right), \tag{9}$$

with the Bessel function I_0. While K_0 is a *fundamental solution* to Eq. (4), I_0 is its associated *Riemann function*. Other convergent contours only yield linear combinations of these functions. Inspecting the asymptotic behavior of G_1 and G_2, one recognises that G_1 diverges along the "lightcone" in the (a, χ)-space, while G_2 diverges exponentially for large arguments. Therefore, neither of these two solutions can be used to construct wave packets following the classical solutions. This is an important result, since it demonstrates that the relation between no-boundary solutions and the classical theory is very loose. My conjecture is that these properties hold in general. The asymptotic behavior does not change if the condition $a' = \chi' = 0$ is relaxed.

The path integral, when evaluated exactly, does therefore exhibit properties that cannot necessarily be seen in a semiclassical approximation. This has consequences, for example, for the discussion of the arrow of time in a recollapsing quantum Universe [11].

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3 Quantum Cosmological Path Integral for a Model with a Cosmological Constant

The purpose of this section is to discuss a quantum cosmological model which exhibits certain nontrivial features with respect to path integration in curvilinear coordinates. Configuration spaces with curvature (and torsion) find fruitful application, for example, in quantum mechanics with a Coulomb potential [1,12]. For our purpose it is sufficient to consider a one-dimensional model in which the Wheeler-DeWitt equation (2) is given by

$$\hat{H}\psi(\alpha) = \frac{1}{2}\left(\frac{d^2}{d\alpha^2} - e^{4\alpha} + \lambda^2 e^{8\alpha}\right)\psi(\alpha) = 0 \ . \tag{10}$$

Here, $\alpha = \ln a$, where a is the scale factor, and λ^2 denotes a positive cosmological constant. It should be noted that the term arising from general relativity would read $\lambda^2 e^{6\alpha}$; however, the choice in Eq. (10) enables an exact evaluation of the path integral, without changing the qualitative features of the realistic model. It is also possible to obtain the potential used in Eq. (10) from general relativity, but this would require a very unrealistic equation of state [13].[a]

Due to the presence of the complicated potential in Eq. (10), a direct evaluation of the path integral (3) is impossible. One can, however, try to perform a space-time transformation in the path integral to cope with this problem. Such a procedure has turned out to be very useful for evaluating path integrals in situations where a direct calculation is unfeasible. This has been elaborated in Ref. [12], where the hydrogen atom has been mapped into a harmonic-oscillator system, which is why it is now called the Duru-Kleinert transformation. Several other systems have been investigated by this method [1,14,15]. A particular example is the case of the Morse potential which can be mapped into the system of a harmonic oscillator with a centrifugal barrier [1,15]. In fact, the potential in our model (10) is similar to the Morse potential, which is why an analogous transformation can be applied.

Because Eq. (10) resembles the case of Liouville quantum mechanics, the transformation

$$q = e^{2\alpha} \Leftrightarrow \alpha = \frac{1}{2}\ln q \equiv F(q) \tag{11}$$

[a] I am grateful to Alexander Zhuk for pointing this out to me.

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is convenient. Thus one finds for the Hamiltonian

$$\hat{H} = 2\left(q^2\frac{d^2}{dq^2} + q\frac{d}{dq}\right) - \frac{q^2}{2} + \frac{\lambda^2 q^4}{2} . \tag{12}$$

The next step is to introduce a time-transformation in the $D\alpha$ part of the path integral (which is an ordinary quantum mechanical path integral) according to

$$dt = [F'(q)]^2 d\tau = \frac{d\tau}{4q^2(\tau)} , \tag{13}$$

with $\tau(T) \equiv s$. This simplifies both the kinetic and the potential term. One arrives at the new Hamiltonian

$$\tilde{H} = [F'(q)]^2 \hat{H} = \frac{1}{2}\left(\frac{d^2}{dq^2} + \frac{1}{q}\frac{d}{dq}\right) - \frac{1}{8} + \frac{\lambda^2 q^2}{8} . \tag{14}$$

It is shown in the Appendix that the implementation of all these transformations in the path integral leads to the appearance of a quantum correction

$$\Delta V = \frac{1}{8q^2} \tag{15}$$

to the potential. It is called *quantum* because it is proportional to $\hbar^2$ (here set equal to one). The correction appears because the lattice definition of the path integral requires the use of a certain ordering prescription for coordinates and momenta [1], which is given in the present case by Weyl ordering and midpoint prescription (see the Appendix). It should be emphasised, however, that this quantum correction to the potential is a *formal* correction which is needed for a correct evaluation of the path integral; there is *no* correction whatsoever to the potential in the Wheeler-DeWitt equation (10). In fact, the path integral is used to gain solutions (or Green functions) to Eq. (10).

The path integral can then be written as

$$G(q'', q') = \frac{1}{2\sqrt{q'q''}} \int ds \langle q'', s; q', 0\rangle = \frac{1}{2\sqrt{q'q''}} \times$$
$$\int ds \int_{q(0)=q'}^{q(s)=q''} Dq(\tau) \exp\left(\frac{is}{8} + i\int_0^s \left[-\frac{\dot{q}^2}{2} - \frac{1}{8q^2} - \frac{\lambda^2 q^2}{8}\right] d\tau\right) . \tag{16}$$

The origin of the factor $1/2\sqrt{q'q''}$ lies in the demand for covariance with respect to point canonical transformations [1]. An exact expression for

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$\langle q'', s; q', 0\rangle$ can be given if one recalls that the path integral corresponding to the Lagrangian

$$L(x, \dot{x}) = \frac{\dot{x}^2}{2} - \frac{\omega^2 x^2}{2} - \frac{g}{x^2} \tag{17}$$

can be expressed in closed form [16]. The present case corresponds to the choice $g = -1/8$ and $\omega^2 = \lambda^2/4$, remembering that the kinetic term in our model is negative definite. It is interesting that this value for g corresponds to the limiting case in quantum mechanics, where a particle can fall into the centre under the influence of a radial potential. We thus have

$$\langle q'', s; q', 0\rangle = \frac{i\lambda\sqrt{q'q''}}{2\sinh(\lambda s/2)} \exp\left(\frac{is}{8} - \frac{i\lambda}{4}[q'^2 + q''^2]\coth\frac{\lambda s}{2}\right) J_0\left(\frac{-\lambda q' q''}{2\sinh(\lambda s/2)}\right), \tag{18}$$

where J_0 denotes a Bessel function.

For $\lambda \neq 0$, the classical solutions are constrained to regions $\alpha > \alpha_0$, where $\alpha_0 = -(1/2)\ln\lambda$ is the zero point of the potential (there exists also the possibility of the classical system to sit at $\alpha = -\infty$).

Turning to the evaluation of the path integral (16), the integral to be evaluated reads, taking into account Eq. (18),

$$G(q'', q') = i\int \frac{d\tau}{2\sinh\tau} \exp\left(\frac{i\tau}{4\lambda} - \frac{i\lambda}{4}[q'^2 + q''^2]\coth\tau\right) J_0\left(\frac{-\lambda q' q''}{2\sinh\tau}\right), \tag{19}$$

where $\tau = \lambda s/2$ is introduced as the new integration variable. This integral can easily be evaluated by a steepest-descent approximation, similar to the discussion in Ref. [5]. One thereby finds saddle points contributing either a factor $e^{-c/\lambda}$ (saddle points in the upper half plane) or a factor $e^{c/\lambda}$ (saddle points in the lower half plane), where c is positive. If λ were dynamical, this would lead to a "peak" of the wave function either at zero or infinite cosmological constant. Consider now, for the case $\lambda q > 1$, the contour parallel to the real axis of Fig. 1. It can be deformed into a contour (shown as a dashed line) which receives its dominant contributions from the steepest-descent paths through the saddle points C and D. The integral (19) can, however, be evaluated *exactly* along this contour for any value of q' (not necessarily no-boundary conditions). Writing $\tau = \rho + i\pi/2$, one finds

$$G(q'', q') = \frac{1}{2}\exp\left(-\frac{\pi}{8\lambda}\right) \times$$

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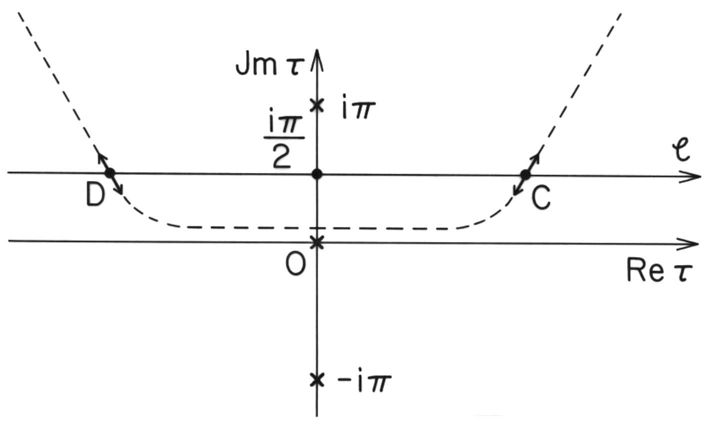

Figure 1. The exact integration contour (solid line) can be deformed into a contour (dashed line) which receives its dominant contributions from steepest-descent contours through the saddle points C and D.

$$\int_{-\infty}^{\infty} \frac{d\rho}{\cosh\rho} \exp\left(\frac{i\rho}{4\lambda} - \frac{i\lambda}{4}[q'^2 + q''^2]\tanh\rho\right) J_0\left(\frac{i\lambda q' q''}{2\cosh\rho}\right) , \qquad (20)$$

which yields the result [17]

$$G(q'', q') = \frac{-i\pi \exp(-\pi/8\lambda)}{\lambda q' q'' \cosh(\pi/8\lambda)} M_{i/8\lambda,0}\left(\frac{i\lambda q'^2}{2}\right) M_{i/8\lambda,0}\left(\frac{i\lambda q''^2}{2}\right) , \qquad (21)$$

where M denotes the Whittaker functions. One can easily see that Eq. (21) satisfies the Wheeler-DeWitt equation (10) for both arguments. The expression in Eq. (21) separates in q' and q'', so there is not much loss of generality in discussing only the case $q' = 0$. In this limit, Eq. (21) becomes

$$G(q) = \frac{\pi \exp(-\pi/8\lambda)}{2\cosh(\pi/8\lambda)} \exp(-i\lambda q^2/4) M\left(\frac{1}{2} - \frac{i}{8\lambda}, 1, \frac{i\lambda q^2}{2}\right) , \qquad (22)$$

with

$$M_{\kappa,\mu}(z) = e^{-z/2} z^{1/2+\mu} M\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right), \qquad (23)$$

which relates Whittaker's function to Kummer's function. It is not surprising that the above expressions resemble the expressions for the energy Green

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function in the quantum mechanical case of the Morse potential [15].

Both the semiclassical approximation to Eq. (22) and the steepest-descent approximation to Eq. (19) yield the same result, namely

$$G(q) \overset{\lambda q \gg 1}{\sim} 2\sqrt{\frac{\pi}{\lambda q^2}} \exp\left(-\frac{\pi}{8\lambda}\right) \cos\left(-\frac{\ln 2\lambda q}{4\lambda} + \frac{\lambda q^2}{4} - \frac{1}{8\lambda} - \frac{\pi}{4}\right) . \tag{24}$$

The factor $\exp(-\pi/8\lambda)$ is the usual WKB penetration factor that *would* correspond to a particle tunnelling from $q = 1/\lambda$ to $q = 0$. The concept "tunnelling" must, however, be used with great care, since no tunnelling process is happening – due to the absence of any external time. This is a major difference between quantum mechanics and quantum cosmology [10,11].

Appendix

The origin of the problem with path integration on curved manifolds (or in curvilinear coordinates) lies in the very definition of the path integral, which is only valid in Cartesian coordinates. The occurrence of a quantum correction ΔV to the potential is directly connected with the ordering prescription of the lattice coordinates and momenta in the action, which in turn is related to operator ordering in the Hamiltonian [1].

For a quantum mechanical system whose classical counterpart is given by the Lagrangian

$$L(q,\dot{q}) = \frac{m}{2} g_{ab}(q)\dot{q}^a\dot{q}^b - V(q) , \tag{25}$$

the corresponding Hamilton operator is Weyl ordered, if the *midpoint* prescription is used in the path integral [18], i.e. if the lattice action is taken to be of the form

$$S = \sum_{k=1}^{N} \left(\frac{m}{2\epsilon} g_{ab}(\bar{q}_k)(q_k^a - q_{k-1}^a)(q_k^b - q_{k-1}^b) - \epsilon V(\bar{q}_k)\right) , \tag{26}$$

where

$$\bar{q}_k \equiv \frac{q_k + q_{k-1}}{2} . \tag{27}$$

The Hamilton operator is Weyl ordered, if it is of the form

$$H_W = \frac{1}{8m}(g^{ab}p_a p_b + 2p_a g^{ab} p_b + p_a p_b g^{ab}) + V(q) , \tag{28}$$

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and it is obtained by randomly ordering all coordinates and momenta, counting all different orderings once and forming their arithmetic mean.

The natural ordering for the Hamiltonian, however, consists in the choice of the Laplace-Beltrami operator Δ_{LB} for the kinetic term, because this preserves the covariance of the theory. In fact, this is what one arrives at, when performing a transformation from Cartesian to curvilinear coordinates. We thus consider the Hamiltonian

$$H = -\frac{1}{2}\Delta_{LB} + V(q) = \frac{1}{2}g^{-1/4}p_a g^{ab} g^{1/2} p_b g^{-1/4} + V(q) = H_W + \Delta V \ , \quad (29)$$

where g is the determinant of the metric g_{ab}. To evaluate the matrix element $\langle q'|\exp(-iHt)|q\rangle$ with H according to Eq. (29), we thus have to take into account the correction ΔV in the action (26). The canonical momenta are given by

$$p_a = \frac{1}{i}\left(\frac{\partial}{\partial q^a} + \frac{1}{2}\frac{\partial \ln\sqrt{g}}{\partial q^a}\right) \ , \quad (30)$$

because they are in this form – as well as the Hamiltonian (29) – self-adjoint with respect to the inner product

$$\langle\varphi|\psi\rangle = \int \ dq\sqrt{g}\varphi^*(q)\psi(q) \ . \quad (31)$$

By comparing Eq. (28) and Eq. (29), one finds an explicit expression for ΔV,

$$\Delta V = \frac{1}{8m}(g^{ac}\Gamma^d_{ac}\Gamma^b_{bd} - R) \ , \quad (32)$$

where Γ^c_{ab} and R are the Christoffel symbols and the Ricci scalar of the configuration space, respectively.

In the quantum cosmological model discussed in Section 3, we can easily find ΔV directly. Starting from Eq. (14), we first recognise that due to the time transformation (13) the kinetic term of $\tilde{H}$ is not the Laplace-Beltrami operator with respect to any metric. The procedure described above has therefore to be slightly modified. The Hamiltonian $\tilde{H}$ is self-adjoint with respect to an inner product of the form (31), where $\sqrt{g}$, however, must be changed into

$$J(q) = \exp\left(\int \frac{dq}{q}\right) = q, \quad (33)$$

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arising from the first-derivative term in Eq. (14).[b] The canonical momentum then reads according to Eqs. (30) and (33)

$$p_q = -i\left(\frac{d}{dq} + \frac{1}{2q}\right) , \tag{34}$$

leading to

$$\tilde{H} = -\frac{p_q^2}{2} + \frac{1}{8q^2} - \frac{1}{8} + \frac{\lambda^2 q^2}{8} . \tag{35}$$

Due to the trivial form of the second-derivative term in Eq. (14), this is already Weyl ordered, and ΔV can be directly read off, leading to the expression (15). Finally, I would like to mention that all these transformations can be implemented consistently into the measure of the path integral, leading in fact to the expression (16) with the correct prefactor.

Acknowledgments

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References

[1] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995).
[2] D. Giulini, *Phys. Rev. D* **51**, 5630 (1995); D. Giulini and C. Kiefer, *Phys. Lett. A* **193**, 21 (1994); H.-J. Schmidt, in *Differential Geometry and Its Application*, Eds. J. Janyška and D. Krupka (World Scientific, Singapore, 1990), p. 405.
[3] J.J. Halliwell, in *Conceptual Problems of Quantum Gravity*, Eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991).
[4] J.J. Halliwell and J. Louko, *Phys. Rev. D* **39**, 2206 (1989).
[5] C. Kiefer, *Ann. Phys. (NY)* **207**, 53 (1991).
[6] P. Hájíček, *J. Math. Phys.* **27**, 1800 (1986).
[7] C. Kiefer, in *The Evolution of the Universe*, Eds. G. Börner and S. Gottlöber (John Wiley & Sons, Chichester, 1997).
[8] J.B. Hartle and S.W. Hawking, *Phys. Rev. D* **28**, 2960 (1983).
[9] C. Kiefer, *Class. Quantum Grav.* **10**, S233 (1993).

[b] Alternatively, one can perform a transformation of the wave function in order to arrive directly at a Hamiltonian with standard form, cf. Ref. [19].

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

[10] J. Butterfield and C. Isham, in *The Arguments of Time*, Ed. J. Butterfield (Oxford University Press, Oxford, 1999).
[11] C. Kiefer and H.D. Zeh, *Phys. Rev. D* **51**, 4145 (1995).
[12] I.H. Duru and H. Kleinert, *Phys. Lett. B* **84**, 30 (1979).
[13] C. Stornaiolo, *Phys. Lett. A* **189**, 351 (1994).
[14] C. Grosche and F. Steiner, *Handbook of Feynman Path Integrals* (Springer-Verlag, Berlin, 1998).
[15] A. Pelster, *Zur Theorie und Anwendung nichtintegrabler Raum-Zeit-Transformationen in der klassischen Mechanik und in der Quantenmechanik* (Shaker-Verlag, Aachen, 1996).
[16] D.C. Khandekar and S.V. Lawande, *J. Math. Phys.* **16**, 384 (1975).
[17] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, Orlando, 1980).
[18] T.D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, Chur, 1981).
[19] A. Pelster and A. Wunderlin, *Z. Phys. B* **89**, 373 (1992).