Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

THE AHARONOV-BOHM EFFECT IN FOUR DIMENSIONS

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The path-integral method is particularly useful for studying global questions of topology. We discuss the influence of the Aharonov-Bohm effect in four-dimensional spherical systems via the path integral. As a realization, we give the exact Green's function of the relativistic A-B-C system. The procedure can be generalized to any-dimensional systems.

1 Introduction

It is an honor and a pleasure to present an article at this celebration of Professor Dr. Hagen Kleinert's birthday. During the past several years, I have spent some time studying relativistic systems using path integrals. It was Kleinert who first introduced the space-time transformation in the path integral [2] to compute the hydrogen atom [2–4] leading to a vigorous development in this field. We want to show the consequences of the topological effects of the Aharonov-Bohm (A-B) effect in four-dimensional spherically symmetric systems. We study such systems, since several physical systems turn in momentum space into a dynamical problem of a three-dimensional surface in four dimensions, such as the hydrogen atom whose dynamics in momentum space takes place on a three-dimensional sphere in four-dimensional space [4].

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2 The General Influence of the A-B Effect in Spherical Symmetric Systems

The path-integral representation for the Green's function of a relativistic particle moving in external electromagnetic fields in the four-dimensional space is given by [3,5,6]:

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc}\int_0^\infty dS \int \mathcal{D}\rho(\lambda)\Phi\left[\rho(\lambda)\right]\int \mathcal{D}^4 x(\lambda) \\ \times \exp\left\{-A_E\left[\mathbf{x},\dot{\mathbf{x}}\right]/\hbar\right\}\rho(0), \tag{1}$$

where the Euclidean action reads

$$A_E\left[\mathbf{x},\dot{\mathbf{x}}\right] = \int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{m}{2\rho\left(\lambda\right)}\dot{\mathbf{x}}^2\left(\lambda\right) - i(e/c)\mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}}(\lambda) \right. \\ \left. -\rho(\lambda)\frac{\left(E - V(\mathbf{x})\right)^2}{2mc^2} + \rho\left(\lambda\right)\frac{mc^2}{2}\right], \tag{2}$$

with S defined as

$$S = \int_{\lambda_a}^{\lambda_b} d\lambda\rho(\lambda). \tag{3}$$

Here $\rho(\lambda)$ is an arbitrary dimensionless fluctuating scale variable, $\rho(0)$ is the terminal point of the function $\rho(\lambda)$, and $\Phi[\rho(\lambda)]$ is some convenient gauge-fixing functional [3,5,6]. The only condition on $\Phi[\rho(\lambda)]$ is that

$$\int \mathcal{D}\rho(\lambda)\Phi\left[\rho(\lambda)\right] = 1. \tag{4}$$

The prefactor $\hbar/mc$ in Eq. (1) is the Compton wave length of a particle of mass m, and $\mathbf{A}(\mathbf{x})$, $V(\mathbf{x})$ stand for the vector and scalar potential of the system, respectively. The constant E is the system energy, and $\mathbf{x}$ is the spatial part of the $(4+1)$-vector $x^\mu = (\mathbf{x}, \tau)$. The functional integral for $\mathbf{x}$ in the representation of Eq. (1) can be interpreted as the expectation value of the real functional $\exp\left\{-\int_{\lambda_a}^{\lambda_b} d\lambda\beta\rho(\lambda)V(\mathbf{x}(\lambda))/\hbar\right\}$ over the measure

$$K_0(\mathbf{x}_b, \mathbf{x}_a; \lambda_b - \lambda_a) = \int \mathcal{D}^4 x(\lambda)\exp\left\{ -\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda \right. \\ \left. \times \left[\frac{m}{2\rho\left(\lambda\right)}\dot{\mathbf{x}}^2\left(\lambda\right) - i\frac{e}{c}\mathbf{A}(x)\cdot\dot{\mathbf{x}}(\lambda) - \rho\left(\lambda\right)\frac{V^2(\mathbf{x})}{2mc^2}\right]\right\}. \tag{5}$$

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With this interpretation, the entire Green's function turns into the following formula

$$
\begin{aligned}
G(\mathbf{x}_b,\mathbf{x}_a;E) = \frac{i\hbar}{2mc}\int_0^\infty dS \int \mathcal{D}\rho(\lambda)\Phi\left[\rho(\lambda)\right] e^{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda)\mathcal{E}} \\
\times \left\langle \exp\left\{ -\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda \beta\rho(\lambda)V(\mathbf{x}(\lambda)) \right\} \right\rangle \rho(0),
\end{aligned} \tag{6}
$$

in which $\mathcal{E} = (m^2c^4 - E^2)/2mc^2$ and $\beta = E/mc^2$, with the notation $\langle\star\rangle$ standing for the expectation value of the moment $\star$ over the measure $K_0(\mathbf{x}_b,\mathbf{x}_a;\lambda_b-\lambda_a)$. Equation (6) forms the basis for studying the relativistic potential problems via the Feynman-Kac type formula.

Expanding the potential $V(\mathbf{x})$ in Eq. (6) into a power series and interchanging the order of integration and summation, we have

$$
\begin{aligned}
G(\mathbf{x}_b,\mathbf{x}_a;E) = \frac{i\hbar}{2mc}\int_0^\infty dS \int \mathcal{D}\rho\Phi\left[\rho\right] e^{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda)\mathcal{E}} \\
\times \sum_{n=0}^\infty \frac{(-\beta/\hbar)^n}{n!}\left\langle \left(\int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda)V(\mathbf{x}(\lambda)) \right)^n \right\rangle \rho(0).
\end{aligned} \tag{7}
$$

We see that the calculation of the relativistic path integral now turns into the computation of the expectation value of moments Q^n ($Q = \int_{\lambda_a}^{\lambda_b} d\lambda\rho V(\mathbf{x})$) over the Feynman measure and their summation in accordance with the Feynman-Kac type formula. Ordering the λ as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_b$ and denoting $\mathbf{x}(\lambda_i) = \mathbf{x}_i$, the perturbation series in Eq. (7) explicitly turns into [1]

$$
\begin{aligned}
\sum_{n=0}^\infty \frac{(-\beta/\hbar)^n}{n!}\left\langle \left(\int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda)V(\mathbf{x}(\lambda)) \right)^n \right\rangle = K_0(\mathbf{x}_b,\mathbf{x}_a;\lambda_b-\lambda_a) \\
+ \sum_{n=1}^\infty \left(-\frac{\beta}{\hbar}\right)^n \int_{\lambda_a}^{\lambda_b} d\lambda_n \int_{\lambda_a}^{\lambda_n} d\lambda_{n-1}\cdots\int_{\lambda_a}^{\lambda_2} d\lambda_1 \\
\times \int \left[\prod_{j=0}^n K_0(\mathbf{x}_{j+1},\mathbf{x}_j;\lambda_{j+1}-\lambda_j)\right]\prod_{i=1}^n \rho_i V(\mathbf{x}_i)d\mathbf{x}_i,
\end{aligned} \tag{8}
$$

where $\lambda_0 = \lambda_a, \lambda_{n+1} = \lambda_b, \mathbf{x}_{n+1} = \mathbf{x}_b$, and $\mathbf{x}_0 = \mathbf{x}_a$. The merit of expanding the path integral into this form is that we can let $\Phi[\rho]$ be equal to the delta functional $\delta[\rho - 1]$ in order to fix the value of $\rho(\lambda)$ to unity, such that the integration over S in Eq. (7) leads to a Laplace transformation [7–9]. Because

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of the convolution property of the Laplace transformation, we obtain

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc}\Bigg\{ G_0(\mathbf{x}_b, \mathbf{x}_a; \mathcal{E}) + \sum_{n=1}^{\infty} \left(-\frac{\beta}{\hbar}\right)^n \int \left[\prod_{j=0}^{n} G_0(\mathbf{x}_{j+1}, \mathbf{x}_j; \mathcal{E})\right] \prod_{i=1}^{n} V(\mathbf{x}_i) d\mathbf{x}_i \Bigg\}, \tag{9}$$

where $G_0(\mathbf{x}_b, \mathbf{x}_a; \mathcal{E})$ is the Laplace transformation of the pseudo propagator $K_0(\mathbf{x}_b, \mathbf{x}_a; \lambda_b - \lambda_a)$. For calculating the explicit form of the Green's function G_0, the relations between the Cartesian coordinates (x_1, x_2, x_3, x_4) and the polar coordinates $(r, \vartheta, \theta, \varphi)$, with the range of validity $0 < r < \infty$, $0 < \vartheta, \theta < \pi$, and $0 < \varphi < 2\pi$, read

$$\begin{aligned} &x_1 = r\cos\vartheta, \quad x_2 = r\sin\vartheta\cos\theta, \quad x_3 = r\sin\vartheta\sin\theta\cos\varphi, \\ &x_4 = r\sin\vartheta\sin\theta\sin\varphi. \end{aligned} \tag{10}$$

With these transformations, the volume element is

$$d\mathbf{x} = r^3 \sin^2\vartheta \sin\theta dr d\vartheta d\theta d\varphi. \tag{11}$$

We have the angular decomposition of the spherical symmetric systems

$$G_0(\mathbf{x}_{j+1}, \mathbf{x}_j; \mathcal{E}) = \sum_{s=0}^{\infty}\sum_{l=0}^{s}\sum_{k=-l}^{l} g_l^{(0)}(r_{j+1}, r_j; \mathcal{E}) Y_{slk}(\hat{\mathbf{x}}_{j+1}) Y_{slk}^*(\hat{\mathbf{x}}_j), \tag{12}$$

where $Y_{slk}(\hat{\mathbf{x}}_j) = Y_{slk}(\vartheta_j, \theta_j, \varphi_j)$ is the four-dimensional spherical harmonics. Using the orthogonality relations,

$$\int \sin^2\vartheta \sin\theta dr d\vartheta d\theta d\varphi Y_{slk}(\hat{\mathbf{x}}) Y_{s'l'k'}^*(\hat{\mathbf{x}}) = \delta_{ss'}\delta_{ll'}\delta_{mm'}, \tag{13}$$

the intermediate angular part can be performed. It yields

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc}\sum_{s=0}^{\infty}\sum_{l=0}^{s}\sum_{k=-l}^{l} G_l(r_b, r_a; \mathcal{E}) Y_{slk}(\hat{\mathbf{x}}_b) Y_{slk}^*(\hat{\mathbf{x}}_a), \tag{14}$$

with the pure radial Green's function

$$G_l(r_b, r_a; \mathcal{E}) = \sum_{n=0}^{\infty}\left(-\frac{\beta}{\hbar}\right)^n g_l^{(n)}(r_b, r_a; \mathcal{E}). \tag{15}$$

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The radial integrations are given by

$$g_l^{(n)}(r_b, r_a; \mathcal{E}) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=0}^{n} g_l^{(0)}(r_{j+1}, r_j; \mathcal{E}) \right] \prod_{i=1}^{n} r_i^3 V(r_i) dr_i. \tag{16}$$

Here the unperturbed radial Green's function $g_l^{(0)}$ has the path-integral representation

$$g_l^{(0)}(r_{j+1}, r_j; \mathcal{E}) = \frac{1}{r_{j+1} r_j} \int_0^\infty dS e^{-\mathcal{E}S/\hbar} \int \mathcal{D}r(\lambda)$$
$$\times \exp\left\{ -\frac{1}{\hbar} \int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{m}{2} \dot{r}^2(\lambda) - i\frac{e}{c} \mathbf{A}(x) \cdot \dot{\mathbf{x}}(\lambda) - \frac{V^2(r)}{2mc^2} \right] \right\}. \tag{17}$$

It is now that we can consider the influence of the A-B effect for systems with spherical symmetry. The vector potential of the A-B effect is given by

$$\mathbf{A}(\mathbf{x}) = 2g \frac{-x_4 \hat{e}_3 + x_3 \hat{e}_4}{x_3^2 + x_4^2}, \tag{18}$$

where $\hat{e}_{3,4}$ stands for the unit vector along the x, y axis. Introducing the azimuthal angle around the A-B tube

$$\varphi(\mathbf{x}) = \arctan(x_4/x_3), \tag{19}$$

the interaction from the A-B effect, denoted in the action by the subscript mag, has the form

$$A_{\text{mag}} = i\hbar\mu_0 \int_0^S d\lambda \dot{\varphi}(\lambda), \tag{20}$$

where $\varphi(\lambda) = \varphi(\mathbf{x}(\lambda))$, $\dot{\varphi} = d\varphi/d\lambda$, and $\mu_0 = -2eg/\hbar c$ is a dimensionless number. The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in space-time can be considered as being closed at infinity, and the integral

$$k = \frac{1}{2\pi} \int_0^S d\lambda \dot{\varphi}(\lambda) \tag{21}$$

is a topological invariant with integer values of the winding number k. The magnetic interaction is therefore purely topological, its value being

$$A_{\text{mag}} = i\hbar\mu_0 2k\pi. \tag{22}$$

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The influence of the A-B effect in the entire Green's function $G(\mathbf{x}_b, \mathbf{x}_a; \mathcal{E})$ can be considered by writing down the explicit form of the four-dimensional spherical harmonics

$$\begin{aligned}\sum_{s=0}^{\infty}\sum_{l=0}^{s}\sum_{k=-l}^{l} & Y_{slk}(\hat{\mathbf{x}}_b)Y^*_{slk}(\hat{\mathbf{x}}_a) \\ &= \sum_{s=0}^{\infty}\sum_{l=0}^{s}\sum_{k=-l}^{l}\frac{2^{2l+1}(s+1)\Gamma^2(l+1)\Gamma(s-l+1)}{\pi\Gamma(s+l+2)}\left(\sin\vartheta_b\sin\vartheta_a\right)^l \\ &\quad \times C^{l+1}_{s-l}(\cos\vartheta_b)C^{l+1}_{s-l}(\cos\vartheta_a)Y_{lk}(\theta_b,\varphi_b)Y^*_{lk}(\theta_a,\varphi_a), \end{aligned} \tag{23}$$

where C^{λ}_n and Y_{lk} are the Gegenbauer polynomials and the usual three-dimensional spherical harmonics. The orthogonality relations of $Y_{slk}(\hat{\mathbf{x}})$ can be easily checked by the formula:

$$\int_0^{\pi} d\vartheta \sin^{2\lambda}\vartheta C^{\lambda}_m(\cos\vartheta)\, C^{\lambda}_n(\cos\vartheta) = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!\,(\lambda+n)\,\Gamma^2(\lambda)}\delta_{m,n}. \tag{24}$$

With the help of the following relation between the associated Legendre polynomial $P^{\mu}_{\nu}(x)$ and the Jacobi function $P^{(\alpha,\beta)}_n(x)$ [8],

$$P^k_l(\cos\theta) = (-1)^k\frac{\Gamma(1+k+l)}{\Gamma(1+l)}\left(\cos\theta/2\sin\theta/2\right)^k P^{(k,k)}_{l-k}(\cos\theta), \tag{25}$$

the angular part of Eq. (23) turns into

$$\begin{aligned}\sum_{s=0}^{\infty}\sum_{l=0}^{s}\sum_{k=-l}^{l} & Y_{slk}(\hat{\mathbf{x}}_b)Y^*_{slk}(\hat{\mathbf{x}}_a) \\ &= \sum_{s=0}^{\infty}\sum_{l=0}^{s}\sum_{k=-l}^{l}\frac{2^{2l+1}(s+1)(2l+1)\Gamma(s-l+1)\Gamma(l+k+1)\Gamma(l-k+1)}{4\pi^2\Gamma(s+l+2)} \\ &\quad \times \left(\sin\vartheta_b\sin\vartheta_a\right)^l C^{l+1}_{s-l}(\cos\vartheta_b)C^{l+1}_{s-l}(\cos\vartheta_a)e^{ik(\varphi_b-\varphi_a)} \\ &\quad \times \left(\cos\theta_b/2\cos\theta_a/2\sin\theta_b/2\sin\theta_a/2\right)^k P^{(k,k)}_{l-k}(\cos\theta_b)P^{(k,k)}_{l-k}(\cos\theta_a). \end{aligned} \tag{26}$$

To go further, let us change the variables s, l by defining $s-l=t$, $l-k=q$ into t, q. Thus the Green's function of Eq. (14) becomes

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc}\sum_{t=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{2(q+k)+1}G_{q+k}(r_b, r_a; \mathcal{E})$$

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$$
\begin{aligned}
&\times \frac{(t+q+k+1)\,[2(q+k)+1]\,\Gamma(q+2k+1)\Gamma(q+1)\Gamma(t+1)}{4\pi^2\Gamma(t+2(q+k)+2)} \\
&\times (\sin\vartheta_b \sin\vartheta_a)^{q+k}\, C_t^{q+k+1}(\cos\vartheta_b) C_t^{q+k+1}(\cos\vartheta_a) e^{ik(\varphi_b-\varphi_a)} \\
&\times (\cos\theta_b/2\cos\theta_a/2\sin\theta_b/2\sin\theta_a/2)^k\, P_q^{(k,k)}(\cos\theta_b) P_q^{(k,k)}(\cos\theta_a). \quad (27)
\end{aligned}
$$

Now, let us consider the A-B effect by invoking Poisson's summation formula

$$
\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} dy \sum_{n=-\infty}^{\infty} e^{2\pi n y i} f(y). \tag{28}
$$

The entire Green's function $G_0(\mathbf{x}_b, \mathbf{x}_a; \mathcal{E})$ containing the A-B effect becomes

$$
\begin{aligned}
G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2mc} \sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \int dz \sum_{k=-\infty}^{\infty} 2^{2(q+z)+1} G_{q+z}(r_b, r_a; \mathcal{E}) \\
&\times \frac{(t+q+z+1)\,[2(q+z)+1]\,\Gamma(q+2z+1)\Gamma(q+1)\Gamma(t+1)}{4\pi^2\Gamma(t+2(q+z)+2)} \\
&\times (\sin\vartheta_b \sin\vartheta_a)^{q+z}\, C_t^{q+z+1}(\cos\vartheta_b) C_t^{q+z+1}(\cos\vartheta_a) e^{i(z-\mu_0)(\varphi_b+2k\pi-\varphi_a)} \\
&\times (\cos\theta_b/2\cos\theta_a/2\sin\theta_b/2\sin\theta_a/2)^z\, P_q^{(z,z)}(\cos\theta_b) P_q^{(z,z)}(\cos\theta_a). \quad (29)
\end{aligned}
$$

The sum over all k in Eq. (29) forces z to be equal to μ_0 modulo an arbitrary integral number leading to

$$
\begin{aligned}
G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2mc} \sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{2(q+|k+\mu_0|)+1} G_{q+|k+\mu_0|}(r_b, r_a; \mathcal{E}) \\
&\times \frac{[2(q+|k+\mu_0|)+1]\,\Gamma(q+2\,|k+\mu_0|+1)\Gamma(q+1)\Gamma(t+1)}{4\pi^2\Gamma(t+2(q+|k+\mu_0|)+2)} \\
&\times (t+q+|k+\mu_0|+1)\,(\cos\theta_b/2\cos\theta_a/2\sin\theta_b/2\sin\theta_a/2)^{|k+\mu_0|} \\
&\times (\sin\vartheta_b \sin\vartheta_a)^{q+|k+\mu_0|}\, C_t^{q+|k+\mu_0|+1}(\cos\vartheta_b) C_t^{q+|k+\mu_0|+1}(\cos\vartheta_a) \\
&\times P_q^{(|k+\mu_0|,|k+\mu_0|)}(\cos\theta_b) P_q^{(|k+\mu_0|,|k+\mu_0|)}(\cos\theta_a) e^{ik(\varphi_b-\varphi_a)}. \quad (30)
\end{aligned}
$$

From this equation, we see that the influence of the A-B flux will cover any dimensions although it is just described by the two-dimensional coordinates. Its effects are not only in the energy spectra but also in the wave functions of higher-dimensional spaces. As a realization, we consider the pure Coulomb system moving in the fields of A-B in four dimensions. Since the potential of Coulomb is $-e^2/r$, the pure radial amplitude $G_{q+|k+\mu_0|}(r_b, r_a; \mathcal{E})$ has the

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form

$$G_{q+|k+\beta_0|}(r_b, r_a; \mathcal{E}) = \frac{m}{\hbar}\frac{1}{r_b r_a}\sum_{n=0}^{\infty}\left(\frac{m\beta e^2}{\hbar^2}\right)^n g^{(n)}_{q+|k+\mu_0|}(r_b, r_a; \mathcal{E}), \tag{31}$$

with

$$g^{(n)}_{q+|k+\beta_0|}(r_b, r_a; \mathcal{E}) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=0}^{n} g^{(0)}_{q+|k+\beta_0|}(r_{j+1}, r_j; \mathcal{E})\right]\prod_{i=1}^{n} dr_i, \tag{32}$$

in which $g^{(0)}_{q+|k+\beta_0|}$ is given by [7]

$$\begin{aligned} g^{(0)}_{q+|k+\beta_0|}&(r_{j+1}, r_j; \mathcal{E}) \\ &= \int_0^\infty \frac{dS}{S} e^{-\frac{\mathcal{E}}{\hbar}S} e^{-m(r_{j+1}^2+r_j^2)/2\hbar S} I_{\sqrt{[(q+|k+\mu_0|)+1]^2-\alpha^2}}\left(\frac{m}{\hbar}\frac{r_{j+1}r_j}{S}\right). \end{aligned} \tag{33}$$

To obtain the explicit result of $g^{(n)}_{q+|k+\beta_0|}$, we note that [7]

$$\begin{aligned} \int_0^\infty & \frac{dS}{S} e^{-\frac{\mathcal{E}}{\hbar}S} e^{-m(r_b^2+r_a^2)/2\hbar S} I_\rho\left(\frac{m}{\hbar}\frac{r_b r_a}{S}\right) \\ &= 2\int_0^\infty dz \frac{1}{\sinh z} e^{-\kappa(r_b+r_a)\coth z} I_{2\rho}\left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z}\right) \end{aligned} \tag{34}$$

with $\kappa = \sqrt{m^2c^4 - E^2}/\hbar c$. With the help of the integral formula

$$\int_0^\infty dr r e^{-r^2/a} I_\nu(\varsigma r) I_\nu(\xi r) = \frac{a}{2} e^{a(\xi^2+\varsigma^2)/4} I_\nu\left(a\xi\varsigma/2\right), \tag{35}$$

we obtain the result

$$\begin{aligned} g^{(1)}_{q+|k+\beta_0|}(r_b, r_a; \mathcal{E}) &= \int_0^\infty g^{(0)}_{q+|k+\beta_0|}(r_b, r; \mathcal{E}) g^{(0)}_{q+|k+\beta_0|}(r, r_a; \mathcal{E}) dr \\ &= \frac{2^2}{\kappa}\int_0^\infty z h(z) dz, \end{aligned} \tag{36}$$

where the function $h(z)$ is defined as

$$h(z) = \frac{1}{\sinh z} e^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{[(q+|k+\beta_0|)+1]^2-\alpha^2}}\left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z}\right). \tag{37}$$

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The expression for $g^{(n)}_{q+|k+\beta_0|}(r_b, r_a; \mathcal{E})$ can be obtained by induction with respect to n, and is given by

$$g^{(n)}_{q+|k+\beta_0|}(r_b, r_a; \mathcal{E}) = \frac{2^{n+1}}{n!} \frac{1}{\kappa^n} \int_0^\infty z^n h(z) dz. \tag{38}$$

Inserting this expression in Eq. (31), we obtain

$$\begin{aligned} G_{q+|k+\beta_0|}(r_b, r_a; \mathcal{E}) = &\frac{m}{\hbar} \frac{2}{r_b r_a} \int_0^\infty dz e^{\left(\frac{2m\beta e^2}{\hbar^2 \kappa}\right) z} \\ &\times \frac{1}{\sinh z} e^{-\kappa(r_b + r_a)\coth z} I_{2\sqrt{[(q+|k+\beta_0|)+1]^2 - \alpha^2}} \left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z} \right). \end{aligned} \tag{39}$$

The integration can be done by using the formula

$$\begin{aligned} \int_0^\infty dy \frac{e^{2\nu y}}{\sinh y} \exp\left[-\frac{t}{2} (\zeta_a + \zeta_b) \coth y \right] I_\mu \left(\frac{t\sqrt{\zeta_b \zeta_a}}{\sinh y} \right) \\ = \frac{\Gamma\left((1+\mu)/2 - \nu \right)}{t\sqrt{\zeta_b \zeta_a} \Gamma(\mu+1)} W_{\nu,\mu/2}(t\zeta_b) M_{\nu,\mu/2}(t\zeta_a), \end{aligned} \tag{40}$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are the Whittaker functions. We complete the integration and obtain

$$\begin{aligned} G_{q+|k+\mu_0|}(r_b, r_a; \mathcal{E}) = &\frac{1}{(r_b r_a)^{3/2}} \frac{mc}{\sqrt{m^2c^4 - E^2}} \\ &\times \frac{\Gamma\left(1/2 + \sqrt{[(q+|k+\mu_0|)+1]^2 - \alpha^2} - E\alpha/\sqrt{m^2c^4 - E^2} \right)}{\Gamma\left(1 + 2\sqrt{[(q+|k+\mu_0|)+1]^2 - \alpha^2} \right)} \\ &\times W_{E\alpha/\sqrt{m^2c^4-E^2}, \sqrt{[(q+|k+\beta_0|)+1]^2 - \alpha^2}} \left(\frac{2}{\hbar c} \sqrt{m^2c^4 - E^2} r_b \right) \\ &\times M_{E\alpha/\sqrt{m^2c^4-E^2}, \sqrt{[(q+|k+\beta_0|)+1]^2 - \alpha^2}} \left(\frac{2}{\hbar c} \sqrt{m^2c^4 - E^2} r_a \right). \end{aligned} \tag{41}$$

Inserting this in Eq. (30), we obtain the entire relativistic Green's function of the A-B-C system in four-dimensional space. The energy spectrum is determined by the poles of the Gamma function

$$1/2 + \sqrt{[(q+|k+\mu_0|)+1]^2 - \alpha^2} - E\alpha/\sqrt{m^2c^4 - E^2} = -n_r, \tag{42}$$

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where $n_r = 0, 1, 2, \ldots$. It is easy to find the energy levels by expanding the parameter α in the lowest two orders:

$$E_{q,k} = mc^2 \left\{ 1 - \frac{\alpha^2}{2\left(n_r + q + |k + \mu_0| + 3/2\right)^2} - \frac{\alpha^4}{\left(n_r + q + |k + \mu_0| + 3/2\right)^3} \right. \\ \left. \times \left[\frac{1}{2q + 2\,|k + \mu_0| + 2} - \frac{3}{8} \frac{1}{\left(n_r + q + |k + \mu_0| + 3/2\right)} \right] + \mathcal{O}(\alpha^6) \right\}. \qquad (43)$$

We see that, if the flux is quantized, the result is reduced to the pure relativistic Coulomb spectrum in four dimensions [7] by $4\pi g = 2\pi\hbar c/e\times$ integer, leading to an integer-valued $|k + \mu_0|$. In comparison with the conventional path-integral treatment of the hydrogen atom, the merit of our approach is that the complicated multi-valued Kustaanheimo-Stiefel and the space-time transformation techniques are avoided. Furthermore, the calculation of the entire Green's function is just related to the unperturbed one.

3 Discussion

We have discussed the influence of the A-B effect on the single particle system moving in a spherical symmetric potential. We have shown that the influence of the flux on the energy spectra and wave functions will cover any space dimensions. This is not common for particles moving in magnetic fields. We remind the Laudau levels of a particle moving in a homogeneous magnetic field $\mathbf{B}$ pointing along the direction of the third axis. The magnetic field can be described by the vector potential $\mathbf{A}(\mathbf{x}) = (0, Bx, 0)$ or a more complicated one by performing phase transformation of an appropriate gauge. For the exact propagator it turns out that the particle is free in the third axis and, of course, in higher dimensions. The reason lies in the topological effect of the A-B coupling to the angular momentum, leading to effects in any dimension. In recent years, topological effects and geometrical phases have penetrated many areas of science. We hope that our discussion contributes to our understanding of the role of topology in physics.

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