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CRITICAL BEHAVIOR OF CORRELATION FUNCTIONS AND ASYMPTOTIC EXPANSIONS OF FEYNMAN AMPLITUDES

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We present a connection between the critical behavior of correlation functions and the general theory of asymptotic behaviors of Feynman amplitudes. Using the Mellin representation of Feynman integrals, an asymptotic expansion for a generic Feynman amplitude can be obtained for any set of invariants going to zero or to ∞. If we take all masses going to zero in Euclidean metric, the truncated expansion has a rest compatible with the convergence of the series. In analogy to the application of field theory to critical phenomena, we consider from our general asymptotic expansions the critical behavior of correlation functions, in particular the critical behavior of the two-point function.

1 Introduction

In field theory, a special situation arises for Euclidean Green's functions in the momentum representation, when vanishingly small values for the external momenta are considered besides the zero-mass limit for the fields. In this case, we speak of the infrared (divergent) behavior of correlation functions. These divergences, which are seen as a "pathological" behavior in the context of applications of field theories to particle physics, are associated with the large-distance correlations in statistical systems and play a crucial role in the study of critical phenomena and phase transitions in such systems.

In this note we adopt a mathematical physicist's point of view, in the framework of perturbative field theory, starting from the observation that infrared (critical) behaviors of correlation functions in Euclidean field theories

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may be seen as a special case of a general class of asymptotic behavior of Feynman amplitudes, as some of the involved masses tend to zero. The use of the perturbative method can be justified in applications of field theory to critical phenomena, for the examples of models of field theory that have been found to give relevant informations. These informations are controlled by the free field fixed-point, or by fixed-points that approach the free field fixed-point in some limit. This means that the Feynman diagram approach to field theory plays an important role in understanding physical situations in critical phenomena. As we have stressed above, the large-distance correlations in statistical systems are particularly important, as they play a crucial role in the study of phase transitions. In field theory language, these large-distance correlations manifest themselves as infrared behaviors of correlation functions, which are in perturbative language a particular case of asymptotic behaviors of Feynman amplitudes. This is one of the reasons why the analysis presented in this note could be interesting for the perturbative field theoretical approach to critical phenomena. For a complete account on the application of field theoretical methods to critical phenomena the reader is referred to the books by Kleinert [1].

Divergent large-distance behaviors of renormalized field theories containing massless fields and infrared divergences received a large amount of attention over the last decades. Historically, in applications to particle physics, they have been considered as an indesirable feature, a kind of “illness” of the theory which should be “cured” at any price. Actually these divergences appear at different levels. For Green’s functions in Minkowskian metric it has been shown a long time ago that for some theories (e.g. QED_4) Green’s functions exist at the zero-mass limit for some particles, as distributions on the 4-momenta. This means that Green’s functions are well defined quantities in the infrared limit [2]. For particles on mass shell, Green’s functions generally do not have a limit for those theories, even if they are well defined off mass shell. The oldest and best known examples are infrared divergences in scattering amplitudes in QED. Since the work of Bloch and Nordsieck [3], this problem has been investigated exhaustively [4,5].

It is worthwhile to emphasize that, in contrast to what happens in applications to particle physics, in applications of field theory to critical phenomena both ultraviolet divergences and infrared behaviors need not to be “cured”. The ultraviolet cutoff is related to the inverse of some fundamental length of the system such as the atomic scale and the infrared behaviors of correlation functions describe directly the approach to critical points.

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We make use of Mellin transform techniques to represent Feynman integrals, along similar lines as it has been done to study renormalization and asymptotic behaviors of scattering amplitudes [6–8], and to study the heat kernel expansion [9]. To fix our framework we consider a theory involving scalar fields $\varphi_i(x)$ having masses m_i, defined on a Euclidean space. For simplicity we may think of a single scalar field $\varphi(x)$ having a mass m. A generic Feynman graph G is a set of I internal lines, L loops, q connected components (a graph is disconnected if $q > 1$) and n vertices linked by some (polynomial) potential. To each vertex are attributed external momenta $\{p_i\}$ and internal ones $\{k_a\}$. A subgraph $S \subset G$ is defined as a graph, where all lines, vertices, and loops belong to G and a quotient graph G/S is a graph obtained from G reducing S to a point. A $q - tree$ of the diagram G is a subgraph of G having q connected components, without loops and linking all vertices of G. The cases $q = 1$ $(1 - trees)$ and $q = 2$ $(2 - trees)$ are of particular interest for us.

The Feynman amplitude $G(\{a_k\})$ corresponding to the diagram G is a function of the set of invariants $\{a_k\}$ built from external momenta, $\sum_i p_i^2$, and squared masses m_i^2; it is defined in the Schwinger-Bogoliubov representation by [2,10]

$$G(a_k) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}}(\alpha) e^{-\frac{V(\alpha)}{U(\alpha)}}, \tag{1}$$

where D is the Euclidean space dimension with a positive metric.

In the above formula, the Symanzik polynomials $U(\alpha)$ and $V(\alpha)$ are constructed from the graph G by the prescription

$$U(\alpha) = \sum_{1.T} \prod_{i \notin 1.T} \alpha_i \tag{2}$$

and

$$V(\alpha) = \sum_{2.T} \left(\sum p_j\right)^2 \left(\prod_{i \notin 2.T} \alpha_i\right) + \left(\sum_{j \in G} m_j^2 \alpha_j\right) U(\alpha), \tag{3}$$

where the symbols $\sum_{1.T}$ and $\sum_{2.T}$ mean summation over the $1 - trees$ and $2 - trees$ of G, respectively. The sum $\sum p_j$ in Eq. (3) is the total external momentum entering one of the $2 - tree$ connected components (any one of them equivalently, by momentum conservation). Notice that $U(\alpha)$ and $V(\alpha)$

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are homogeneous polynomials in the α-variables, of degrees L and $L+1$, respectively.

2 Mellin Representation and Asymptotic Expansions of Feynman Amplitudes

In the following we have in mind the physical situation of the infrared behavior, but we would like to emphasize that our method is quite general, in the sense that it applies to any asymptotic limit in Euclidean metric (any choice of the subset a_l below), for arbitrarily given external momenta, generic or exceptional, and for arbitrary vanishing or finite masses. If we perform a scale transformation on the subset $\{a_l\}$ of invariants, $a_l \to \lambda a_l$, the polynomial V splits into two parts,

$$V(\lambda a_m) = \lambda W(a_l, \alpha) + R(a_q, \alpha)\,, \tag{4}$$

where the polynomials $W(a_l, \alpha)$ and $R(a_q, \alpha)$ are also homogeneous of degree $L+1$ in the α-variables.

To be concrete we consider here a special situation with the external momenta $\{p_i\}$ fixed and we investigate the limit $\lambda \to 0$ corresponding to vanishing masses. In this case W is just the second term in Eq. (3). As we have noted above, the method applies along the same lines to any other class of asymptotic behavior. Indeed we note that from a dimensional argument,

$$G\left(\frac{a_l}{\lambda}, a_q\right) = \lambda^{\omega} G(a_l, \lambda a_q), \tag{5}$$

where $\omega = I - DL/2$. This means that the study of a given subset going to zero is equivalent to study the $\lambda \to \infty$ limit on the complementary subset of invariants.

Under the λ-scaling performed in Eq. (4) G becomes a function of λ, $G(\lambda)$, and its Mellin transform, $M(z) = \int_0^{\infty} d\lambda \lambda^{-z-1} G(\lambda)$ may be written in the form

$$M(z) = \Gamma(-z) \int_0^{\infty} \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} \left(\frac{W}{U}\right)^{z}. \tag{6}$$

The scaled amplitude $G(\lambda)$ associated to the Feynman graph G may be obtained by the inverse Mellin transform,

$$G(\lambda) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} dz \lambda^{z} M(z), \tag{7}$$

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where $\sigma = \mathrm{Re}\,(z) < 0$ belongs to the analyticity domain of $M(z)$.

Since the integrand of Eq. (7) vanishes exponentially at $\sigma \pm i\infty$, due to the behavior of $\Gamma(z)$ at large values of $\mathrm{Im}\,z$, the integration contour may be displaced to the right by Cauchy's theorem, picking up successively the poles of the integrand, provided we can desingularize the integral in Eq. (6). Such a problem has been studied by an appropriate choice of local coordinates [11] and also using Hepp sectors and a multiple Mellin representation [6].

From these works it has been possible to show that $M(z)$ has a meromorphic structure of the form

$$M(z) = \sum_{n,q} \frac{A_{nq}\, q!}{(z-n)^{q+1}}. \tag{8}$$

It results from the displacement of the integration contour in the inverse Mellin transform, an expansion for small values of λ, of the form

$$G(\lambda) = \sum_{n=n_0}^{N} \lambda^n \sum_{q=0}^{q_{\max}(n)} A_{nq} ln^q(\lambda) + R_N(\lambda), \tag{9}$$

where the coefficients $A_n(\{p\})$ and the powers of logarithms come from the residues at the poles $z = n$.

The rest of the expansion $R_N(\lambda)$ is given by

$$R_N(\lambda) = \int_{-\infty}^{+\infty} \frac{d(\mathrm{Im}\,z)}{2i\pi} \lambda^z \Gamma(-z) F(z), \tag{10}$$

with

$$N < \mathrm{Re}\,(z) < N+1, \quad \mathrm{Re}\,(z) = N + \eta, \quad 0 < \eta < 1 \tag{11}$$

and where

$$F(z) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} \left(\frac{W}{U}\right)^z. \tag{12}$$

It is a rather difficult task to perform explicitly the α-integrations in Eq. (6) above for a general Feynman amplitude. As this calculation will not be necessary for our purposes, we give the appropriate references for the interested reader [6,7,12,13]. It is shown in these papers that renormalized Feynman amplitudes can be expressed as finite sums of convergent integrals which are exactly of the same type as those of convergent diagrams, provided the various integration variables associated to the remainders of renormalization Taylor

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operators are renamed as supplementary Hepp-sector variables. In the following we keep the notations corresponding to convergent graphs, which means that the results are valid for convergent as well as for renormalized divergent diagrams.

We have shown the convergence of asymptotic expansions of general Feynman amplitudes in another article [14], obtaining a bound for the remainder of the expansion. In the particular case of all masses going to zero (infrared behavior), we have shown that for $I - DL/2 > 0$ (which is just the condition for UV convergence for the dimensionally regularized amplitude), the rest of the asymptotic expansion may be written in the form

$$|R_N(\lambda)| < K_1 K_N(\{p\})(\mu^2)^N \lambda^N, \tag{13}$$

where K_1 and K_N are finite constants and μ is a finite mass scale. The scaling parameter λ is arbitrarily small in the limit of vanishing masses. Therefore the factor $(\lambda\mu^2)^N$ in the bound above makes the sequence of the remainders $R_N(\lambda)$ converge to zero as $N \to \infty$, which is a condition for the convergence of the asymptotic expansion.

3 The 2-Point Function Critical Behavior

In another article we have shown [14] that one can obtain a convergent series from Eq. (9) as $N \to \infty$. Let us specify to the limit of all masses going to zero, and consider for simplicity the case of a single field having mass m. The analysis below can be generalized without difficulty to the case of several fields having different masses. We also consider dimensionally regularized amplitudes, that is, we take the Euclidean space dimension D to be such that the amplitudes are formally defined as convergent integrals, divergences appearing later as singularities for some diagrams.

For the 2-point function $G^{(2)}(p^2, m^2)$, the only nonzero invariant of the type $(\sum_i p_i)^2$ contributing to the construction of the Symanzik polynomial $V(\alpha)$ in Eq. (3) is p^2. This may be seen if we note that for any diagram G contributing to the two-point function, the whole set of two-trees in the definition of $V(\alpha)$ in Eq. (3) divides into two classes, in which the total external momentum entering one of its connected components is either p^2 or zero (named respectively *relevant* and *irrelevant* two-trees). In this case, after introducing a fixed mass scale μ, it is easy to see that the inverse Mellin transform, Eq. (7), may be rewritten in terms of the variable p^2/m^2. Then the small mass behavior of a graph G contributing to the two-point function,

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$G^{(2)}(p^2, m^2)$, has the form

$$G(p^2, m^2) = \sum_{n=-\omega}^{\infty} \left(\frac{m^2}{p^2}\right)^n \sum_{q=0}^{q_{\max}(n)} A_{nq}(\mu^2) \left[-\ln\left(\frac{m^2}{p^2}\right)\right]^q . \tag{14}$$

We remember that according to Eqs. (6) and (7) the expansion above comes from the inverse Mellin transform,

$$G\left(\frac{m^2}{p^2}\right) = \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} dz \left(\frac{p^2}{m^2}\right)^z \Gamma(-z) \times \int_0^{\infty} \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\mu^2 \sum_{i\in G} \alpha_i} \left(\frac{W'}{U}\right)^z , \tag{15}$$

where $W' = \mu^2(\sum_{j\in G} \alpha_j) U(\alpha)$, $R'(\alpha) = \sum'_{2.T} \prod'_{i\notin 2.T} \alpha_i$, and the notations $\sum'$ and $\prod'$ indicate respectively summation and product over *relevant* two-trees.

The coefficients $A_{nq}(\mu^2)$ in Eq. (14) come from the meromorphic structure of the Mellin transform displayed in Eq. (8). It is an extremely hard task to determine explicitly all these coefficients, which is equivalent to completely desingularize the integral over the α-variables in Eq. (15) respective to z. The coefficients $A_{\omega q}$ in Ref. [10] and Ref. [6] corresponding to the leading poles of the Mellin transform have been studied in the case of the behavior of Feynman amplitudes at large momenta, which is mathematically equivalent to the case of vanishing masses studied here. We adapt the method used in the above mentioned works to get an expression for the leading coefficients in the case of the small mass behavior. In the following we give only the general lines of the method, the results we have obtained and the definitions of the basic objects. The calculations are very involved and the full mathematical details in the case of large momenta behavior are in the above quoted references. These calculations can be adapted to our case without major difficulties. The main tool used to perform the analytic continuation of the Mellin transform is the generalized Taylor operator, a generalization of the operators used in field theory for the purpose of renormalization. It is defined as follows: given a function $f(x)$, such that $x^{-\nu} f(x)$ is infinitely differentiable at $x = 0$, we define the generalized Taylor operator τ^n as

$$\tau_x^n f(x) = x^{-\lambda-\epsilon} T^{n+\lambda} [x^{\lambda+\epsilon} f(x)], \tag{16}$$

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where T is the usual Taylor operator, $\lambda \geq -E'(\nu)$ is an integer, $E'(\nu)$ is the smallest integer $\geq \mathrm{Re}(\nu)$ and $\epsilon = E'(\nu) - \nu$. For any subdiagram $S \subseteq G$ this corresponds to a generalized Taylor operator defined by

$$\tau_S^n f(\{\alpha\}) = [\tau_\varrho^n f(\{\alpha\})|_{\alpha_i = \varrho^2 \alpha_i, i \in S}]_{\varrho=1}. \tag{17}$$

A basic quantity associated to the diagram G playing a role in the desingularization procedure has the form

$$\prod_{S \subseteq G} \left[1 - \tau_S^{I(S)}\right] \left\{ U^{\frac{D}{2}}(\alpha) \left[\frac{W'(\alpha)}{U(\alpha)}\right]^z \right\}, \tag{18}$$

where the product runs over subdiagrams S of the graph G, including G itself. Although the τ operators do not commute, it can be shown that the complete product $\prod_{S \subseteq G}(1 - \tau_S^{I(S)}[.])$ is independent of the order of application of the factors upon the function between the brackets, $[.]$.

The procedure follows along lines parallel as was done in Ref. [10] and in Ref. [6] for the large momenta behavior. We obtain for the leading coefficients $A_{\omega q}$ the expression

$$\begin{aligned} A_{\omega q} = \frac{1}{q!} \sum_{\{S_1, \ldots, S_k\}} & \frac{2^k}{(k-q-1)!} \xi_{G/S_1} \tilde{\xi}_{S_1/S_2} \ldots \tilde{\xi}_{S_{k-1}/S_k} \\ & \times \left[\frac{d^{k-q-1}}{dz^{k-q-1}} [\Gamma(-z) \tilde{\gamma}_{S_k}(z)]\right]_{z=-n_0}. \end{aligned} \tag{19}$$

In the above equation, ω is the Weinberg leading power

$$\omega = \mathrm{Sup}_G\, [\omega(S)], \tag{20}$$

where Sup_G runs over the superficial degree of divergence of all essential subdiagrams of G, $\omega(S) = L(S)D - 2I(S)$ and $q_{\mathrm{max}}(\omega) = Q - 1$, Q being the number of elements in the largest set of nested leading subdiagrams. The sum runs over all forests $\{S_1, ..., S_k\}$ of $k(> q)$ nested *leading* subdiagrams $S_1 \supset S_2 \supset ... \supset S_k$ (we remember that *leading* subdiagrams are those whose superficial degree of divergence equals ω). The quantities ξ, $\tilde{\xi}$, $\tilde{\gamma}$ are obtained from the subdiagrams $S \subseteq G$ by the formulas

$$\xi_S = \int_\kappa^\infty \prod_{i \in S} d\alpha_i e^{-\mu^2 \sum_{i \in S} \alpha_i} \prod_{S' \subseteq S} \left[1 - \tau_{S'}^{-2I(S')}\right] U_S^{\frac{D}{2}}(\alpha), \tag{21}$$

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$$\tilde{\xi}_S = -\frac{d\xi_S}{d\mu^2} = \mu^2 \sum_{i\in S} \xi_{S_i}, \tag{22}$$

$$\begin{aligned}\tilde{\gamma_{S_k}} &= \frac{\Gamma(-\frac{z}{2})}{\Gamma(-z)}\mu^2 \int_0^\infty \prod_{i\in S} d\alpha_i e^{-\mu^2 \sum_{j\in S}\alpha_j} \\ &\times \prod_{S'\subseteq S}\left[1-\tau_{S'}^{-2I(S')}\right]\left[\frac{W_S^{'}}{U_S}(\alpha)\right]^{\frac{z}{2}}\left[\sum_{i\in S}\alpha_i\right] U_S^{-\frac{D}{2}}(\alpha),\end{aligned} \tag{23}$$

where S_i is the diagram obtained from S by inserting a two leg vertex (a mass insertion μ^2) on the line i. Particular cases for the quantities in the above equations are $\varrho_{G/G} = 1$ and $\varrho_{G/S} = 0$ if G itself is leading. The Feynman amplitude corresponding to S_i is simply given by

$$G_{S_i} = \int_0^\infty \prod_{i\in S} d\alpha_i\, U^{-\frac{D}{2}}(\alpha)(\alpha_i) e^{-\frac{V(\alpha)}{U(\alpha)}}. \tag{24}$$

The various factors in Eqs. (19), (22), and (23) can be reorganized to write the leading coefficients in the more convenient form

$$\begin{aligned}A_{\omega q} &= \frac{1}{q!}\sum_{\{S_1,\dots,S_k\}} \frac{2^k}{(k-q-1)!} \\ &\times \frac{d^{k-q-1}}{dz^{k-q-1}}\left[\int_0^\infty \prod_{i\in G} d\alpha_i e^{-\mu^2\sum_{i\in G}\alpha_i} A(\alpha;z)\right]_{z=\omega},\end{aligned} \tag{25}$$

where the function $A(\alpha; z)$ is defined by

$$\begin{aligned}A(\alpha;z) &= \Gamma\left(-\frac{z}{2}\right) R_G^{'}\Bigg[U_{G/S_1}^{-\frac{D}{2}} U_{S_1/S_2}^{-\frac{D}{2}} \dots U_{S_{k-1}/S_k}^{-\frac{D}{2}} U_{S_k}^{-\frac{D}{2}} \\ &\times \left[\sum_{S_1/S_2}\alpha_i\right]\dots\left[\sum_{S_{k-1}/S_k}\alpha_i\right]\left[\sum_{S_k}\alpha_i\right](W_{S_k}^{'}/U_{S_k})^{\frac{z}{2}}\Bigg],\end{aligned} \tag{26}$$

and where, taking the convention $S_0 \equiv G$, $R_G^{'}$ is the operator (the order $-2I(T)$ is understood for each τ operator corresponding to a subdiagram T)

$$R_G^{'} = \prod_{l=1}^{k}\left[\prod_{T_l\subseteq S_{l-1}/S_l}(1-\tau_{T_l})\right]\prod_{T\subseteq S_k}(1-\tau). \tag{27}$$

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The operator $R_G^{'}$ does not change the homogeneity properties of the functions upon which it acts. So, remembering the homogeneity properties of the polynomials U and $W^{'}$, and noting that $L(G/S_1)+L(S_1/S_2)+...+L(S_{k-1}/S_k)+L(S_k) = L(G)$, we see from Eq. (26) that $A(\alpha;z)$ is a homogeneous function in the α-variables of degree $L(G)D/2+k+z/2$. Then taking spherical coordinates in α-space we may write from the preceding equations

$$A_{\omega q} = \frac{1}{q!}\sum_{\{S_1,...,S_k\}} \frac{2^{k-1}}{(k-q-1)!}\frac{d^{k-q-1}}{dz^{k-q-1}}\left[\Gamma\left(-\frac{z}{2}\right)\int d\Omega\int_0^\infty d\varrho\, e^{\varrho\mu^2 f(\Omega)} \right.$$
$$\left. \times \varrho^{I-\frac{L(G)D}{2}+k+\frac{z}{2}-1} g(\Omega;z)\right]_{z=\omega}. \tag{28}$$

The integral over ϱ in the equation above may be expressed in terms of the Γ-function, and $f(\Omega)$, $g(\Omega;z)$ are functions of the angular variables Ω depending on the specific topological characteristics of the graph G considered. We get for the leading coefficients $A_{\omega q}$

$$A_{\omega q} = \frac{1}{q!}\sum_{\{S_1,...,S_k\}} \frac{2^{k-1}}{(k-q-1)!}\frac{d^{k-q-1}}{dz^{k-q-1}}\left[\Gamma\left(-\frac{z}{2}\right)\int d\Omega\, g(\Omega;z)\right.$$
$$\left. \times(\mu^2 f(\Omega))^{-(I-\frac{L(G)D}{2}+k+\frac{z}{2})}\Gamma\left(I-\frac{L(G)D}{2}+k+\frac{z}{2}\right)\right]_{z=\omega}. \tag{29}$$

In the very neighbourhood of criticality the contribution of the amplitude G is given by the leading term in the expansion (14) which corresponds to the highest powers of m^2/p^2 and of $-\ln(m^2/p^2)$. This means that for very small values of m^2 we have

$$G(p^2,m^2) \approx A_{\omega Q}\left(\frac{m^2}{p^2}\right)^{-\omega}\left[-\ln\left(\frac{m^2}{p^2}\right)\right]^{Q-1}, \tag{30}$$

where we remember that ω is the Weinberg leading power, given by Eq. (20), and Q is the number of elements in the largest set of leading subdiagrams of G. To get the coefficient $A_{\omega Q}$ in Eq. (30) from Eq. (29), we note that the sum has only one term, corresponding to the nest $\{S_1,...,S_Q\}$ and a zeroth-order derivative. We obtain

$$A_{\omega Q} = \frac{2^{Q-1}}{(Q-1)!}\Gamma\left(-\frac{\omega}{2}\right)g(\Omega;z)\Gamma\left(I-\frac{L(G)D}{2}+k+\frac{z}{2}\right)$$
$$\times\int d\Omega g(\Omega;\omega)\left[\mu^2 f(\Omega)\right]^{-\left[I-\frac{L(G)D}{2}+k+\frac{z}{2}\right]}. \tag{31}$$

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Now, if G itself is leading it does not contribute to the expansion, since in this case $\varrho_{G/S} = 0$ for every $S \subset G$. If G is not leading, $\omega(G) = L(G) - 2I(G) < \omega$, and there exists a $\delta > 0$ such that $\omega = L(G)D - 2I(G) + \delta$. The arguments of the Γ-functions in Eq. (31) above are respectively $-\omega/2 = (2I(G) - L(G)D - \delta)/2$ and $\delta + Q$. Since $\delta > 0$ and $Q \geq 1$, singularities in $A_{\omega Q}$ come from the factor $\Gamma(-\omega/2)$. Thus at fixed space dimension Δ (for instance $\Delta = 3$), if the diagram G has a topological structure such that $L(G)\Delta - 2I(G) + \delta = 2n$, n an integer ≥ 0, the corresponding singularity of the Γ-function above, $\Gamma(-\omega/2)$, should be removed by a renormalization procedure. This may be done, as usual, taking $D = \Delta - \epsilon$, and subtracting the pole at $\epsilon = 0$, leaving some regular function $\Gamma_{\text{Ren}}(\Delta)$. The result in space dimension Δ for the coefficient $A_{\omega Q}$ reads

$$A_{\omega Q}(G) = \frac{2^{Q(G)-1}}{(Q(G)-1)!} \Gamma_{\text{Ren};G}(\Delta) (\mu^2)^{-[\delta + Q(G)]} \Gamma(\delta + Q(G)) \times \int d\Omega g_G(\Omega; \omega) \left[f_G(\Omega)\right]^{-[\delta + Q(G)]} . \tag{32}$$

In the above equation we have displayed explicitly the dependence of the various quantities on the Feynman amplitude G we have considered.

Thus the behavior of the two-point function near criticality is described by an expression having the form

$$G^{(2)}(p^2, m^2) \approx \sum_G A_{\omega Q}(G) \left(\frac{m^2}{p^2}\right)^{-\omega(G)} \left[-\ln\left(\frac{m^2}{p^2}\right)\right]^{Q(G)-1} , \tag{33}$$

where the symbol $\sum_G$ means summation over the whole set of Feynman diagrams contributing to the two-point function. The quantities under the summation symbol can be obtained by explicitly calculating each Feynman diagram G. This result holds for any scalar field theory without derivative couplings.

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