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HIGHER ALGEBRAIC GEOMETRIZATION EMERGING FROM NONCOMMUTATIVITY

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We review the gradual geometrization which has occurred in fundamental physics from the discovery of special relativity in 1905 to the standard model in 1975. After discussing symmetry and ordinary supersymmetry, we introduce internal supersymmetry. Here the even and odd generators correspond to the form-calculus of a gauge theory with spontaneous symmetry breakdown, with the gauge field one-forms occupying the even submatrices and the Higgs fields zero-forms occupying the off-diagonal submatrices. The Grassmann superalgebra is not (super)-abelian and closes on some semi-simple subalgebra. We study two examples: the electroweak $SU(2/1)$, predicting the mass of the Higgs particle around 130 ± 10 GeV, and $P(4R)$ for Riemannian gravity. Internal supersymmetry does not operate on the physical Hilbert space and as a result of non-commutative geometry, the matter fields in its fibres relate to $Z(2)$ gradings other than that of quantum statistics (chirality in our examples).

1 Introduction - and the Physics of Time

Hagen Kleinert is sixty according to *classical* clocks, but this is clearly a misinterpretation of the data. Observations show that Hagen and Annemarie *have not aged at all.* Had they both been born with twins, he with a twin brother, and she with a twin sister, we might have been able to explain our paradox as a complicated extension of the *twin paradox*, conjecturing that we are now facing the travelling twins, who have just returned and replaced the sedentary couple. However, as the Kleinerts never had a twin couple, we must be facing some as yet uncharted and unidentified relativistic effect, perhaps related to some unknown aspect of quantum gravity - coupled à la Penrose

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with the collapse of the state-vector and thus to the Everett-Wheeler Many-Worlds interpretation .... With the effect still shrouded in mystery, I have had to go along and behave as if I believed that Hagen is indeed getting older (wiser he certainly is) and I am happy to dedicate this study to his (classical) sixtieth birthday and wish Annemarie and him many happy returns. I shall do my best to be around when Hagen's classical age is 75, and partake in the next *Festschrift*, but, alas, I cannot make a firm commitment on this matter.

2 Steps in the Geometrization Process

Hagen's contributions to Physics in the last decade have mostly been of a geometrical nature, involving Riemannian manifolds with torsion - whether in 3 dimensions and the physics of materials (transitions between phases, e.g. melting [1]) or in 4 dimensions and issues relating to general relativity [2]. We physicists of the late XXth (and hopefully of the early XXIst) century have enjoyed the aesthetics and symmetries of the *geometrical representation*. Gradually, between 1905 and 1975, it has become the unique language of physics at the fundamental level [3]. I remind the reader that this takeover occurred in the following stages:

(a) Minkowski's 1907-08 geometrical reinterpretation of Einstein's (1905) special theory of relativity, namely of the symmetries of Maxwell's electromagnetism, as identified by Einstein,

(b) The Einstein-Grossmann application and extension of that model (1911) in a program aiming at reconciling Newtonian mechanics with the above symmetries of electromagnetism, leading to Einstein's construction of the general theory of relativity (GR, 1915) as the new (and fully geometrical) theory of gravity,

(c) The construction of a *gauge theory*, started in 1918 with H. Weyl's (failed) first attempt at a theory of electromagnetism (based on the assumption of local scale-invariance) unifiable with Einstein's gravity (i.e. geometrical). It was followed by his 1928 successful version, in which the geometry is that of a *fibre bundle*, with fibre group $U(1)$ realizing local invariance under transformations of the complex phase angle (introduced by quantum mechanics). This was then generalized (1953) to non-abelian groups by C.N. Yang and R.L. Mills and applied (1975) to the $S[U(2) \otimes U(3)]$ fibre group of the standard model. The latter emerged, on the one hand, as a result of our (1961) $SU(3)_{\text{flavor}}$ classification of the hadrons and the subsequent (1962-64) discovery of the structural mechanism to which it is due, namely the *quark*

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model, followed first (1964-1972) by the introduction of (global) $SU(3)_{\text{color}}$ for the sake of preservation of Fermi-type quantum statistics and then by quantum chromodynamics (QCD), namely *local* $SU(3)_{\text{color}}$, after the discovery (1973) of *asymptotic freedom*. The geometrical nature of local gauge theories was emphasized (1974) by C.N. Yang and T.T. Wu.

(d) Two other developments (also launched in the early twenties) were suggested in the context of *further unification*:

I. Adding dimensions - the program suggested by T. Kaluza (1921) and by O. Klein (1924).

II. Following Einstein, adding an *antisymmetric piece to the metric or connection*.

Developments in the seventies in the unification program, namely in $N_s > 1$ supergravity and in superstring (or "M") theory have converged on a fusion of both these features. After the establishment of relativistic quantum field theory by its success (1948) in quantum electrodynamics (QED), the gravitational field of GR, representing the "fabric" of space-time, should by itself be treated as a quantum field of Bose-type. This thereby does not allow a role for an antisymmetric metric by the spin-statistics theorem. The necessary conditions, however, are induced through (1971-73) *supersymmetry, which adds fermionic degrees of freedom to any boson*. Gravity thereby becomes embedded in supergravity, with the antisymmetric characteristics of (II) represented by the presence of torsion. Finiteness considerations, namely the cancellation of chiral and dilational anomalies, then impose specific higher dimensionalities as in (I). The two programs – torsion and Kaluza-Klein dimensionalities – are thus presently actively pursued in the context of the 11-dimensional $N_s = 1$ *supergravity* constructed by E. Cremmer and B. Julia (1978), reducible in 4-dimensions to the $N_s = 8$ *maximal* or *saturated* supergravity, a version of supergravity which has been shown to represent the low-energy quantum field theory limit of "M-theory", the *state of the art* theory of post-Planck level and quantized gravity. [N_s is the "number of supersymmetries", i.e. the dimensionality of the internal degree of freedom, if any, carried by the Lorentz spinor multiplets of supersymmetry generators].

(e) An independent additional geometrical entry is due to Jean Thierry-Mieg. His 1979 thesis and related articles [4,5] identify the *ghost fields* of a Yang-Mills gauge theory – as conceived by R.P. Feynman (1962) in order to guarantee *off mass shell unitarity* and further developed by B.S. De Witt, L.D. Faddeev, and V.N. Popov – with *odd* elements of the *form calculus*, the

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Grassmann "supercommutative" superalgebra of the gauge theory. Moreover, the "BRST" constraining superalgebra, linking together physical and ghost fields is seen to coincide with the *structural equations* of the fibre bundle.

(f) The superalgebraic system I describe in the rest of this article derives from the latter. It consists in *a nonsupercommutative* extended *superalgebra of forms*, defined over a fibre-bundle *whose base-space is split by a* $Z(2)$ *grading* but which does not necessarily coincide with the $Z(2)$ of quantum statistics (in my examples, it will be the $Z(2)$ of *chirality*). The first model of this type was discovered (1979) by the present author [6] and independently by David Fairlie [7] and involved *the simple Lie supergroup* $SU(2/1)$ *as an "internal supersymmetry", an irreducible algebraic extension of electroweak unification's (spontaneously-broken) local -* $SU(2)\times U(1)$ *symmetry.* The theory has been applied [8] to predict the mass of the Higgs meson, yielding $m(H) = 2m(W)$ in the exact (and unrenormalized) limit, while the inclusion of renormalization effects, as observed in couplings [9], yields as final result $m(H) = 130 \pm 15$ GeV. Note that 9 "events" have been observed at CERN in the fall of 2000 with a Higgs meson mass around 115 GeV.

3 Superalgebras, Supermatrices, and $Z(2)$-Gradings

I first remind the reader of the main definitions and results relating to Lie [10] and to Grassmann (super) algebras [11]. The first involve the application of a $Z(2)$-grading on the basis of the Lie superalgebra as a linear vector space; the $g(x)$ eigenvalue also determines the nature of the $Z(2)$-graded super-Lie bracket $[x, y\}$ and of the relevant super-Jacobi identity. Let the variable $E = \sqrt{1}$ represent the two elements of the finite group $Z(2)$. The superalgebra splits into two subspaces, labelled by that grading,

$$\begin{aligned}
E &= \sqrt{1},\ g = \log_{-1}(E \in Z(2)_g)\,,\\
L &= L_0 + L_1\,,\\
g(x \in L_0) &= 0,\ g(y \in L_1) = 1\,,\\
g([x, y\}) &= g(x) \oplus g(y) \cong 2\,,\\
[x, y\} &= -(-1)^{g(x)\cdot g(y)}[y, x\}\,,\\
[x,\ [y, z\} &= [x, y\},\ z\} + (-1)^{g(x)g(y)}\ [y,\ [x, z\}\}\,.
\end{aligned} \tag{1}$$

In some cases, there also exists a Z-grading $z(L_a) \in Z$, where z is a "quantum number" which is additively preserved by the super-Lie-bracket, though the

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nature of that bracket itself is still determined by the $Z(2)_z$-binary grading within Z, $Z \supset Z(2)_z$,

$$L = \sum_i L^i, \text{ and for any } x \in L^a,\ y \in L^b,\ [x, y\} \subset L^{a+b}\,. \tag{2}$$

Lie superalgebras can always be (and generally are) represented in matrix form, organized in *quarters* Q_g according to

$$\begin{array}{ll} \mid A_0 \mid A_1 \mid & \qquad \mid v^0 \mid \\ \mid B_1 \mid B_0 \mid, & \qquad \mid v^1 \mid, \end{array} \tag{3}$$

the $g = 0$ and $g = 1$ generators thus spanning the squares along or off the main diagonal, respectively. With this supermatrix acting on a column-vector V split in two by some $Z(2)_v$, the $g = 0$ quarters are the endomorphisms of V, namely $A_0 =$ End $(V^0), B_0 =$ End (V^1), whereas the $g = 1$ quarters represent the homomorphisms between the two sectors in V, namely $A_1 =$ Hom $(V_1,\ V_0)$, $B_1 =$ Hom $(V_0,\ V_1)$. In supersymmetry, the $Z(2)_v$ is again the quantum statistics characteristic and correlates with the statistics of the Hilbert space particles in a supersymmetry study.

In the case of a Grassmann (super-commutative) algebra of *differential forms*, the Z grading and its odd-even partitioning $Z(2)_f \subset Z$, respectively, represent the total count in the applications of the exterior derivative d (or the number of differentials involved as factors), and the odd/even partitioning to which this degree belongs. If applied to differential forms arising in an anholonomic basis or in a supergroup manifold, the $Z(2)$ grading fixes the *exterior* (wedge) product according to the rules,

$$\begin{aligned} dx^a \wedge dx^b &= -(-1)^{g(a)g(b)} dx^b \wedge dx^a\,, \\ F_f &= \textstyle\sum_a dx^{a_1} \wedge dx^{a_2} \wedge \cdots dx^{a_f} \mathcal{F}_{a_1 \cdot a_2 \cdots a_f}(x)\,, \\ F^a_{f_1} \wedge F^b_{f_2} &= (-1)^{(f_1 \cdot f_2 + g(a)g(b))} F^b_{f_2} \wedge F^a_{f_1}\,. \end{aligned} \tag{4}$$

A third category of $Z(2)$ gradings describes the intrinsic Poincaré or Lorentz group $Z(2)_s$-grading of the variables of space-time and its double-covering (spin) and the corresponding exterior-derivative operator, as in the case of the Salam-Strathdee "superspace" of supersymmetry. There would then be a need to characterize algebraic structures by $Z(2)_s$, i.e. yet another $Z(2)$, whose eigenvalues we denote by $s(x)$. As we do not deal here with "classical" [Golfand-Likhtman/Wess-Zumino] supersymmetry, we shall not use the $s(x)$.

We now discuss a coupling between superalgebras, in particular the case in which the Lie superalgebra matrices are valued over Grassmann superalgebras

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of differential forms. In these "directly coupled" superalgebras (we use the direct product symbol), the multiplication is fixed by the definition of the $Z(2)$ grading as the base (-1) logarithm of the elements of $Z(2)$, namely the square-roots of the identity, so that for

$$\begin{aligned} A = A_0 + A_1,\ \ F = F_0 + F_1\,, \\ (A \otimes F)_0 = A_0 \otimes F_0 \oplus A_1 \otimes F_1\,, \\ (A \otimes F)_1 = A_1 \otimes F_0 \oplus A_0 \otimes F_1\,, \end{aligned} \tag{5}$$

and the direct product for matrix-elements in two matrices

$$(a \otimes p)(a' \otimes p') = (-1)^{f(p)g(a')}(aa' \otimes pp')\,, \tag{6}$$

with the sign fixed by the $Z(2)$ eigenvalues of p and a', once a' has to move through p to get to a. In any case, the overall grading h is given by

$$h(a \otimes p) = g(a)f(p) + f^n(y \in r_n) \cong 2\,. \tag{7}$$

The simple and semi-simple Lie superalgebras have been classified by V. Kac [12].

4 The Quillen Superconnection

After I had conceived $SU(2/1)$, Jean Thierry-Mieg and I investigated the possibility that the system of forms (the Grassmann superalgebra) in a Yang-Mills theory, *when extended by Higgs fields (i.e. in cases of spontaneous breaking of local symmetry) might generate a nonsupercommutative (or "nonsuperabelian") superalgebra*. Such an extended Grassmann superalgebra might sometimes happen to coincide with a simple Lie superalgebra [$SU(2/1)$ in the electroweak case, etc.]. The idea was partly triggered by the composition of the Lagrangian in such models, *with a term in the Higgs potential quartic in the Higgs field*: such a term could be reproduced by a Lagrangian quadratic in the curvatures, provided these curvatures be taken for a supergroup, in which the even directions $g = 0$ in the superalgebra's $Z(2)_g$ grading are spanned by the original gauge group, while the Higgs fields span the g-odd directions. At that stage, there was no such rederivation for the remaining part of the Higgs potential, namely the *spontaneous symmetry-breakdown triggering* term, quadratic in the Higgs fields and similar to a mass term - but with the inverted sign. In identifying the Higgs fields themselves with even elements in the Grassmann algebra's form-calculus $Z(2)_f$, we were limited at this stage [13], as we had not dared go beyond Thierry-Mieg's original

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identification of the ghosts as vertical components of the gauge fields, packed into contracted one-forms (in the fibre's direction) and the view in which the Higgs fields are *ghosts of ghosts*, i.e. two-forms, twice vertical. For the group-elements to be fully bosonic and Lorentz-invariant, the parameters would coincide for the even subgroup with its ordinary scalar parameters, while the odd part would have the Lorentz-scalar anticommuting ghosts (one-forms in our geometric interpretation of ghosts and BRST).

More about our $SU(2/1)$ example. The group is homomorphic with $Osp(2/2)$, whose fundamental representation is 4-dimensional and fits the quarks [14]. Moreover, one is allowed to add one constant real number to the diagonal quantum numbers; but if by adding one gets integer values for Y_w and for the electric charges - the matrix reduces to a 3-dimensional one. The group thus "knows" that quarks have fractional charges while leptons carry integer ones. Note also that since $I_w = su(2)$ and $I_w \psi_R = 0$, we have for the supertrace $sTr(I_w^z) = 0$; also, as the electrically-charged leptons or quarks are all massive and thus appear both on the right-chiral and left-chiral eigenstates, we also have $str(Q) = 0$.

Some time later and with a more daring mathematical motivation, D. Quillen [15] *postulated* his theory of the *superconnection*, in which the matrix-elements in the odd ($g(a) = 1$) and even ($g(a) = 0$) submatrices of a superalgebra are valued over the Grassmann supercommutative zero-forms ($f = 0$) and one-forms ($f = 1$), respectively, the intertwined coupling thereby ensuring that *the total grading be odd everywhere*, $t = g + f = 1$. It was shown that the 1979 electroweak $SU(2/1)$ could naturally be recast in this mold [16].

5 Noncommutative Geometry: The Electro-Weak Higher Algebraic Geometrization

The third and last step has consisted in reproducing *the entire Yang-Mills Lagrangian with spontaneous symmetry breakdown* directly from one single invariant; in other words, developing a further generalization which has allowed doing it by squaring one single "curvature", the corresponding generalized two-form. It was provided by a variant of A. Connes' *Noncommutative Geometry* [17]. These further developments have drawn on a generalization of the concept of *parallel transport*, as realized by the application of a (covariant) derivative, namely a derivative plus a connection.

At the same time, it also resolved a seemingly paradoxical feature of the

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original "internal supersymmetry" interpretation, namely under the action of the g-odd generators of the superalgebra, the absence in the particle Hilbert space of *boson to fermion* transitions and vice versa. Instead, Hilbert space has carried some other $Z(2)$ grading, unrelated to quantum statistics - *chirality* in the electroweak case, - so that the endomorphisms induced by the odd generators produce a change of chirality, while the even endomorphisms preserve it.

R. Coquereaux and F. Scheck [18,19] were the first to show that this interesting result - namely the interrelationship between physically different $Z(2)$ groups - one in the vector space upon which the transformations are enacted and one in the superalgebra - could be treated as a development of noncommutative geometry (NCG). It was shown that the 1979 electroweak $SU(2/1)$ could naturally be recast in this mold [18-20].

The new arena is a fibre bundle with a non-simply connected base space, namely a direct product of a two-point space $Z(2)$; the points $(1,-1)$ in this realization of $Z(2)$ are labelled $L\&R$ $B = Z(2) \otimes M(1/,3) = M(1,3)_L \oplus M(1,3)_R$ [21]. The gauge-group is the same over the entire basis, but the two fibres carry different representations: for the $SU(2/1)$ example (we refer the reader to Ref. [14] for the detailed algebraic features of this symmetry), the upper (left-chiral) part of V is an $I_{\text{left}} = 1/2 =$ isodoublet $(\nu^0_{e/L}|e^L)$ with weak hypercharges $Y_w = -1$, whereas for the right-chiral $I_{\text{left}} = 0$, $Y_w = -2$ (the assignments were fixed at the time by application of the Gell-Mann/Nishijima rule, as adapted to weak interactions).

Among several new features in noncommutative geometry, the most relevant one is its generalization to discrete spaces of concepts originally related to differentiable manifolds. One such concept, relevant to our present issue, is *parallel transport* [20]. To move from one point $\nu^L(x_L)$ on the fibre, over the point $x_L \in M_L$ in the left-chiral space, to another point $e^-_L(x'_L)$ on that fibre but over a different position x'_L in the same left-chiral space, we just use the partial derivative ∂_μ for an infinitesimal move by a length $\epsilon^\mu(x)$ and later integrate. To preserve the self-parallelism of the fibre at different places, we add a connection $A_\mu(x_L)$ and use a covariant derivative $D_\mu = \partial_\mu + A_\mu(x_L)$. The same is true for moves on the right-chiral space. In principle, this motion is not different from the usual symmetric case - except for the interface with the chiral matter fields.

What do we do, however, to go from $\nu^L(x \in M_L)$ to $e^R(y =\in M_R)$? Besides using the above means, we also have to "jump" between the two chiral spaces - or better, directly between the relevant fibres over them. This

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is a discrete move and it will be achieved by a finite matrix, in $SU(2/1)$ by μ_6 (same as λ_6 in $SU(3)$), which relates $e^-_L \to {e'}^-_R$. It should be anti-hermitian, so that we define the *matrix-derivative* as $T := i\mu_6$. At the same time, however, we need to perform a discrete change in the fibre itself, i.e. transform $(I_w = 1/2, I^z_w = -1/2, Y_w = -1) \to (I_w = I^z_w = 0, Y_w = -2)$, a task for which an appropriate connection is required. It has to resemble Δ as to its Lorentz properties - i.e. it is a scalar. We also note its quantum numbers in $SU(2/1)$: $I_w = 1/2, Y_w = -1$. This is the Higgs field $\Phi(x)$! Altogether, we shall have yet another new piece in the covariant derivative. In the $SU(2/1)$ internal supersymmetry we have a fibre-bundle with structure group $SU(2/1)$ over a split basis $(M_L \oplus M_R)$ and get the expression for the overall curvature

$$\begin{aligned}\mathcal{R} = d\omega + \omega^2 = dA + A^2 + \Phi^2 + d\Phi + A\Phi + T\Phi = R_{YM} + D\Phi \\ + \text{``}V^{1/2}\text{''}\,,\end{aligned} \tag{8}$$

where we regroup the terms in their traditional setup,

$$R_{YM} = dA + A^2, \quad D\Phi = d\Phi + A\Phi, \quad V = [(\Phi)^2]^2 + [T\Phi]^2\,. \tag{9}$$

Squaring that total curvature with its Clebsch-Gordan coefficients and applying $T^2 = -1$ yields the conventional Weinberg-Salam Hamiltonian

$$\begin{aligned} H_{YM} &= {}^*R_{YM} \wedge R_{YM}\,, \\ H(\Phi)_{\text{kinetic}} &= D\Phi^2\,, \\ H(\phi)_{\text{potential}} &= V_\Phi = -\mu\Phi^2 + \lambda(\Phi)^4\,. \end{aligned} \tag{10}$$

6 Higher Algebraic Geometrization and Riemannian Geometry

One of the macroscopic features of this Universe is its obeying the Riemannian constraint, namely,

$$D_\rho g_{\mu\nu} = Q_{\rho\mu\nu} = 0\,. \tag{11}$$

Following Smolin [22], we have conjectured that this describes the state of affairs at low-energy, arising through the degradation of the basic (high-energy) microscopic state, which is then unconstrained and endowed with more symmetry. Assuming the original and quantum-era Universe to have been *affine* [23-25] we may be able to throw some light on the symmetry-breaking mechanism. We have conjectured [26] that this symmetry breakdown occurred through a mechanism of the same type studied in this article.

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I have found that the Higher Algebraic Geometrization is provided here by the simple superalgebra $p(4,R)$, a "hyper-exceptional" in Kac's list. The algebra of the homogeneous symmetry group $SL(4,R)$ on the tetrad frames will sit in the even quarters, i.e. A_0, B_0 in Eq. (3), of the 8×8 matrices of the defining representation of $P(4,R)$, along the diagonal. $SL(4,R)$ will be in its covariant representation in A_0, in the contravariant in B_0. A_1 will contain the 10 symmetric matrices (out of 16) in $GL(4,R)$ and B_1 will contain the 6 antisymmetric ones.

The matrix-derivative will be given by a *unit matrix* in A_1 (or by a Minkowski metric, depending on the issue) and break $SL(4,R)$ down to $SO(4)$ or $SO(3,1)$, i.e. to Riemannian geometry. To justify the introduction of the matrix-derivative we have to start with a chirality-split base space - but this is precisely what we have when we take a Dirac spinor $(1/2,\ 0)\oplus(0,\ 1/2)$ or a *world spinor* [27-29] with this lowest state. We may now write the full "extended curvature" of $P(4,R)$ - including the matrix-derivative piece. It includes the "SKY" [30-32] quadratic $SL(4,R)$ Lagrangian, the kinetic and gauge terms $D\Phi^+$ and $D\Phi^-$, respectively, for the two Higgs holonomic scalars (one a symmetric tensor in the frame indices, one an antisymmetric), a matrix-derivative generated $T\Phi^-$ which will trigger the spontaneous symmetry breakdown, and a term quadratic in the Higgs fields $\{\Phi^+\Phi^-\}$. I have described the physical effects in detail in Ref. [26], with results fitting the observed low-energy Riemannian system.

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