*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann
(World Scientific, Singapore, 2001).

# PATH INTEGRATION AND COHERENT STATES FOR THE 5D HYDROGEN ATOM

N. ÜNAL

*Akdeniz University, Physics Department, P.K. 510 Antalya, 07058, Turkey*
*E-mail: nunal@pascal.sci.akdeniz.edu.tr*

We transform the five-dimensional hydrogen atom into eight oscillators in a parametric time by adding three new degrees of freedom and by using the Hurwitz transformation. The path integration is then performed over the holomorphic coordinates of the oscillators. The end-point integrations over the extra three degrees of freedom give the coherent states of the five-dimensional hydrogen atom.

## 1 Introduction

In 1926, Schrödinger constructed the coherent states for the harmonic oscillator [1] and later some authors searched to obtain the coherent states for the hydrogen atom [2-7].

In 1979, Duru and Kleinert performed the path integration for the hydrogen atom [8]. They made use of the $SO(4)$ dynamical symmetry of the system and transformed the three-dimensional Kepler problem into four harmonic oscillators in a new parametric time by using the Kustaanheimo-Stiefel transformation [9].

Recently, we discussed the path integral for the hydrogen atom in holomorphic coordinates $a_i$ and $a_i^\dagger$ instead of the physical coordinates $\vec{x}$ and $\vec{p}$ [10]. Since the coherent states are the eigenstates of the lowering operator $a_i$, we showed that the kernel between the holomorphic coordinates governs the time evolution of the coherent states. We also discussed the time evolution of the expectation values of $\langle \vec{x} \rangle$ and showed that they satisfy the Kepler laws and that the dispersions also oscillate with the frequency of the elliptic orbit.

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



The transformation $x_1 + ix_2 = (u_1 + iu_2)^2$ maps the two-dimensional Kepler problem into two harmonic oscillators [11]. The Kustaanheimo-Stiefel transformation maps the three-dimensional Kepler problem into four oscillators and this transformation has been used to derive the path-integral solution of the Kepler problem. Cornish discussed the complex form of the Kustaanheimo-Stiefel transformation and showed that it can be written as $x = u^2$ by using quaternions, without using the nonholonomic constraints for the Kepler problem [12].

The aim of this study is to generalize the path-integral method over the holomorphic coordinates of the 5-dimensional hydrogen atom. In order to generalize the path integration methods for the three-dimensional hydrogen atom we will use the Hurwitz transformation which maps the 5-dimensional Kepler problem into 8 harmonic oscillators [13]. In Section 2 we discuss the mapping between the 5-dimensional hydrogen atom and 8 harmonic oscillators and derive the Lagrangian of the system in holomorphic coordinates. In Section 3 we solve the path integral over the holomorphic coordinates, and derive the coherent states of the five-dimensional hydrogen atom in spherical coordinates.

## 2 Classical Kepler Problem in 5D

The action of the 5-dimensional Kepler problem is

$$A = \int_{t_a}^{t_b} dt \left[ \vec{p} \cdot \frac{d\vec{x}}{dt} - \left( \frac{\vec{p} \cdot \vec{p}}{2m} - \frac{k}{r} \right) \right], \tag{1}$$

where $k = e^2$, and $\vec{p}$, $\vec{x}$, and $r$ are

$$\vec{p} = (p_1, p_2, p_3, p_4, p_5), \quad \vec{x} = (x_1, x_2, x_3, x_4, x_5), \quad r = [\vec{x} \cdot \vec{x}]^{\frac{1}{2}}.$$

This system has 15 conserved quantities: 10 for the angular momentum $\epsilon_{ijk} x_j p_k$ and 5 for the Runge-Lenz vector $A_i$. To use this dynamical symmetry we add new degrees of freedom to the free-particle Lagrangian. Then, Eq. (1) becomes

$$A = \int_{t_a}^{t_b} dt \left[ p_A \cdot \frac{dx_A}{dt} - \left( \frac{p_A \cdot p_A}{2m} - \frac{k}{r} \right) \right], \tag{2}$$

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



where

$$x_A = (\vec{x}, x_6, x_7, x_8)\,, \quad p_A = (\vec{p}, p_6, p_7, p_8)\,.$$

We introduce a new parametric time $\lambda$ and the dimensionless variables $X_A$ and $P_A$ according to

$$\frac{dt}{d\lambda} = r, \tag{3}$$

$$X_A = (2m\,|p_0|)^{\frac{1}{2}}\, x_A, \tag{4}$$

$$P_A = (2m\,|p_0|)^{-\frac{1}{2}}\, p_A. \tag{5}$$

Then the action becomes

$$A = \int_{\lambda_a}^{\lambda_b} d\lambda \left[ P_A \frac{dx_A}{d\lambda} + (-p_0)\,\frac{dt}{d\lambda} - \frac{(2m\,|p_0|)^{\frac{1}{2}}}{2m} R\,(P_A P_A + 1) + k \right].$$

The complex form of the Hurwitz transformation is

$$dX_a = \frac{1}{\sqrt{2}} d\,(X_1 + iX_2) = d\xi_A^* \xi_C + \xi_A^* d\xi_C - d\xi_B \xi_D^* - \xi_B d\xi_D^*,$$

$$dX_b = \frac{1}{\sqrt{2}} d\,(X_3 + iX_4) = d\xi_A^* \xi_D + \xi_A^* d\xi_D + d\xi_B \xi_C^* + \xi_B d\xi_C^*,$$

$$dX_c = \frac{1}{\sqrt{2}} d\,(X_5 + iX_6) = \xi_A d\xi_A^* + \xi_B d\xi_B^* - \xi_C^* d\xi_C - \xi_D^* d\xi_D,$$

$$dX_d = \frac{1}{\sqrt{2}} d\,(X_7 + iX_8) = \xi_B d\xi_A + \xi_A d\xi_B - \xi_D d\xi_C - \xi_C d\xi_D, \tag{6}$$

where

$$\xi_A = \frac{1}{\sqrt{2}}\,(u_1 + iu_2)\,, \quad \xi_B = \frac{1}{\sqrt{2}}\,(u_3 + iu_4)\,,$$

$$\xi_C = \frac{1}{\sqrt{2}}\,(u_5 + iu_6)\,, \quad \xi_D = \frac{1}{\sqrt{2}}\,(u_7 + iu_8)\,.$$

In this transformation $X_6$, $X_7$, and $X_8$ are non-holonomic.

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



The transformations for the momenta are

$$P_{\xi_A} = \sqrt{2}\left[P_{x_a^*}\ \xi_C^* + P_{x_b^*}\ \xi_D^* + P_{x_c^*}\ \xi_A^* + P_{x_d}\ \xi_B\right],$$

$$P_{\xi_B} = \sqrt{2}\left[-P_{x_a}\ \xi_D^* + P_{x_b}\ \xi_C^* + P_{x_c^*}\ \xi_B^* - P_{x_d}\ \xi_A\right],$$

$$P_{\xi_C} = \sqrt{2}\left[P_{x_a}\ \xi_A^* + P_{x_b^*}\ \xi_B^* - P_{x_c}\ \xi_C^* - P_{x_d}\ \xi_D\right],$$

$$P_{\xi_D} = \sqrt{2}\left[-P_{x_a^*}\ \xi_B^* + P_{x_b}\ \xi_A^* - P_{x_c}\ \xi_D^* + P_{x_d}\ \xi_C\right]. \tag{7}$$

By using Eqs. (6) and (7) we write the action as

$$A = \int_{\lambda_a}^{\lambda_b} d\lambda \left[\left(P_A \cdot \frac{dX_A}{d\lambda} - \frac{dP_A}{d\lambda}\cdot X_A\right) - p_0\frac{dt}{d\lambda} - \omega\left(P_\xi P_{\xi^\dagger} + \xi^\dagger\xi\right) + k\right], \tag{8}$$

where $\omega = \left(|p_0|/2m\right)^{1/2}$, and $\xi$ and $P_{\xi^\dagger}$ are the following four-component complex spinors

$$\xi = \frac{1}{2}\begin{pmatrix}\xi_A\\ \xi_B\\ \xi_C\\ \xi_D\end{pmatrix}, \quad P_{\xi^\dagger} = \frac{1}{2}\begin{pmatrix}P_{\xi_A^*}\\ P_{\xi_B^*}\\ P_{\xi_C^*}\\ P_{\xi_D^*}\end{pmatrix}.$$

Here $\xi^\dagger$ and $P_\xi$ are defined as

$$\xi^\dagger = \left(\xi^*\right)^T, \quad P_\xi = \left[\left(P_{\xi^\dagger}\right)^*\right]^T.$$

The coherent states are the eigenstates of the lowering operators of the harmonic oscillators. For this reason we define the holomorphic coordinates of the eight oscillators:

$$a = \frac{1}{2}\begin{pmatrix}\xi^\dagger + ip_\xi\\ \xi + ip_{\xi^\dagger}\end{pmatrix}, \quad a^\dagger = \frac{1}{2}\left(\xi - ip_{\xi^\dagger}, \xi^\dagger - ip_\xi\right). \tag{9}$$

In Eq. (9) $a$ and $a^\dagger$ represent eight-component spinors. Then the Lagrangian of the oscillators becomes

$$L = \frac{1}{2i}\left(\frac{da^\dagger}{d\lambda}a - a^\dagger\frac{da}{d\lambda}\right) - p_0\frac{dt}{d\lambda} - \omega\ a^\dagger a + k, \tag{10}$$

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



where $\omega\, a^\dagger a$ is the Hamiltonian of the eight oscillators in the parametric time $\lambda$. The value of the parametric energy is $k$.

## 3 Path Integration

The kernel of the eight oscillators between the holomorphic coordinates $a_a = a(\lambda_a)$ and $a_b^\dagger = a^\dagger(\lambda_b)$ is

$$K\left(a_b^\dagger, t_b; a_a, t_a\right) = \int \frac{D(-p_0)\,Dt}{2\pi} \int \frac{Da^\dagger Da}{(2\pi i)^8}\, e^{iA}, \tag{11}$$

where $A$ is the action of the system and where we use $\hbar = 1$. The relation between the physical kernel $K(\vec{x}_b, t_b; \vec{x}_a, t_a)$ of the 5D Kepler problem and $K(a_b^\dagger, t_b; a_a, t_a)$ is

$$K(\vec{x}_b, t_b;\ \vec{x}_a, t_a) = \int du_6(b)\, du_7(b)\, du_8(b) \int d\lambda K\left(a_b^\dagger, t_b; a_a, t_a\right).$$

We evaluate $K\left(a_b^\dagger, t_b; a_a t_a\right)$ by using the method given in Ref. [14]. The result is

$$K\left(a_b^\dagger, t_b; a_a t_a\right) = \exp\left[a_b^\dagger e^{-i\omega(\lambda_b - \lambda_a)} a_a - i(4\omega - k)(\lambda_b - \lambda_a)\right]. \tag{12}$$

Since the Hurwitz transformation is double-valued, the kernel is symmetric with respect to the end points:

$$K\left(a_b^\dagger, t_b; a_a t_a\right) = \frac{1}{2}\left\{\exp\left[a_b^\dagger e^{-i\omega(\lambda_b-\lambda_a)} a_a\right] + \exp\left[-a_b^\dagger e^{i\omega(\lambda_b-\lambda_a)} a_a\right]\right\} \times e^{-i(4\omega-k)(\lambda_b-\lambda_a)}.$$

$K\left(a^\dagger(\lambda); \alpha(\lambda_a)\right)$ is the matrix element of the evolution operator between $a^\dagger(\lambda) = a^\dagger$ , $\alpha(\lambda_a) = \alpha$. It gives the time evolution of the eigenstate of the lowering operator $a$ in parametric time:

$$K\left(a^\dagger(\lambda); \alpha\right) = |a(\lambda)\rangle = \frac{1}{2}\left[\exp\left(a^\dagger e^{i\omega\lambda}\right)\alpha + \exp\left(-a^\dagger e^{-i\omega\lambda}\right)\alpha\right]. \tag{13}$$

We expand the exponential in Eq. (13) into the power series of $a_i^\dagger$:

$$|a(\lambda)\rangle = \sum_{n_1,\ldots,n_8=0}^{\infty} \left(\frac{1 + (-1)^{n_1+\cdots+n_8}}{2}\right)$$

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



$$\times \frac{(\alpha_1)^{n_1}}{\sqrt{n_1!}} \cdots \frac{(\alpha_8)^{n_8}}{\sqrt{n_8!}} e^{-i\omega(n_1+\cdots+n_8)}, \tag{14}$$

where $|n_i\rangle$ is the energy eigenstate of the $i$th oscillator.

The coherent state of the five-dimensional Kepler problem is obtained by integrating over the final values of the additional free particle coordinates $X_6(b)$, $X_7(b)$, and $X_8(b)$. These integrals are performed in spherical coordinates. We represent the complex coordinates $\xi_A$, $\xi_B$, $\xi_C$, and $\xi_D$ in terms of spherical harmonics as

$$\begin{aligned}
\xi_A &= |u| \cos\frac{\Theta}{2}\left(D_{--}D'_{--} + D_{-+}D'_{+-}\right), \\
\xi_B &= |u| \cos\frac{\Theta}{2}\left(-D_{+-}D'_{--} - D_{++}D'_{+-}\right), \\
\xi_C &= |u| \sin\frac{\Theta}{2}\left(D_{++}D'_{--} - D_{-+}D'_{+-}\right), \\
\xi_D &= |u| \sin\frac{\Theta}{2}\left(D_{+-}D'_{--} - D_{--}D'_{+-}\right),
\end{aligned}$$

where $D_{mm'}$ and $D'_{nn'}$ are

$$D_{mm'} = D^{\left(\frac{1}{2}\right)}_{m/2,m'/2}(\phi,\theta,\psi), \quad D'_{nn'} = D^{\left(\frac{1}{2}\right)}_{n/2,n'/2}(\alpha,\beta,\gamma).$$

Then we can write the $\left\langle \xi,\xi^\dagger \middle| \alpha(\lambda) \right\rangle$ as

$$\left\langle \xi,\xi^\dagger \middle| \alpha(\lambda) \right\rangle = e^{-|\xi|^2+\sqrt{2}\left(\xi\alpha+\xi^\dagger\alpha^\dagger\right)-\alpha\cdot\alpha^\dagger}, \tag{15}$$

where $\alpha$ and $\alpha^\dagger$ are the complex spinors

$$\alpha = (\alpha_-,\beta_-,\gamma_-,\delta)_- = \left(\frac{\alpha_1 - i\alpha_2}{\sqrt{2}}, \frac{\alpha_3 - i\alpha_4}{\sqrt{2}}, \frac{\alpha_5 - i\alpha_6}{\sqrt{2}}, \frac{\alpha_7 - i\alpha_8}{\sqrt{2}}\right),$$

$$\alpha^\dagger = \begin{pmatrix} \alpha_+ \\ \beta_+ \\ \gamma_+ \\ \delta_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 + i\alpha_2 \\ \alpha_3 + i\alpha_4 \\ \alpha_5 + i\alpha_6 \\ \alpha_7 + i\alpha_8 \end{pmatrix}.$$

The expressions $\alpha\cdot\xi$ and $\xi^\dagger\cdot\alpha^\dagger$ are

$$\begin{aligned}
\alpha\cdot\xi &= |u|\left(A_- D'_{--} + B_- D'_{+-}\right), \\
\alpha^\dagger\cdot\xi^\dagger &= |u|\left(A_+ {D'_{--}}^* + B_+ {D'_{+-}}^*\right),
\end{aligned}$$

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



with

$$\begin{aligned} A_- &= \left[ (\alpha - D_{--} - \beta_- D_{+-}) \cos\frac{\Theta}{2} + (\gamma_- D_{++} + \delta_- D_{+-}) \sin\frac{\Theta}{2} \right], \\ B_- &= \left[ (\alpha - D_{--} - \beta_- D_{+-}) \cos\frac{\Theta}{2} + (-\gamma_- D_{++} - \delta_- D_{--}) \sin\frac{\Theta}{2} \right], \\ A_+ &= \left[ \left(\alpha_+ - D^*_{--} - \beta + D^*_{+-}\right) \cos\frac{\Theta}{2} + \left(-\gamma_+ D^*_{-+} + \delta_+ D^*_{+-}\right) \sin\frac{\Theta}{2} \right], \\ B_+ &= \left[ \left(\alpha_+ D^*_{-+} - \beta_+ D^*_{++}\right) \cos\frac{\Theta}{2} + \left(-\gamma_+ D^*_{-+} - \delta_+ D^*_{--}\right) \sin\frac{\Theta}{2} \right]. \end{aligned}$$

We substitute $\alpha \cdot \xi$ and $\alpha^\dagger \cdot \xi^\dagger$ into Eq. (15) and expand the exponential into power series of $D'_{--}$, $D'_{+-}$, ${D'_{--}}^*$, and ${D'_{+-}}^*$:

$$\begin{aligned} \left\langle \xi, \xi^\dagger \middle| \alpha \right\rangle = e^{-|\xi|2} - \alpha \cdot \alpha^\dagger \sum_{n_1;\ldots,n_4=0}^{\infty} |u|^{(n_1+n_2+n_3+n_4)} \left(D'_{--}\right)^{n_1} \left(D'_{+-}\right)^{n_2} \\ \times \frac{(A_-)^{n_1} (B_-)^{n_2} (A_+)^{n_3} (B_+)^{n_4}}{n_1!\, n_2!\, n_3!\, n_4!} \left({D'_{--}}^*\right)^{n_3} \left({D'_{+-}}^*\right)^{n_4}. \end{aligned}$$

Using the following identities

$$\begin{aligned} \left(D'_{--}\right)^{n_1} &= D^{n_1/2}_{-n_1/2,-n_1/2}(\alpha,\beta,\gamma), \\ \left(D'_{+-}\right)^{n_2} &= D^{n_2/2}_{+n_2/2,-n_2/2}(\alpha,\beta,\gamma), \end{aligned}$$

we combine $\left(D'_{--}\right)^{n_1}$ and $\left(D'_{+-}\right)^{n_2}$ as

$$\begin{aligned} \left(D'_{--}\right)^{n_1} \left(D'_{+-}\right)^{n_2} = \left\langle \frac{n_1+n_2}{2}, -\frac{n_1-n_2}{2} \middle| \frac{n_1}{2}, -\frac{n_1}{2}; \frac{n_2}{2} + \frac{n_2}{2} \right\rangle \\ \times D^{(n_1+n_2)/2}_{-(n_1+n_2)/2,-(n_1-n_2)/2}(\alpha,\beta,\gamma). \end{aligned}$$

Then $\langle \xi, \xi^\dagger | \alpha \rangle_{\text{phys}}$ becomes

$$\begin{aligned} \left\langle \xi, \xi^\dagger \middle| \alpha \right\rangle_{\text{phys}} &= e^{-|\xi|^2 - \alpha^\dagger \alpha} \sum_{\substack{n_1,n_2,\\ n_3,n_4=0}}^{\infty} \frac{|u|^{(n_1+n_{12}+n_3+n_4)}}{n_1! \cdots n_4!} (A_-)^{n_1} (B_-)^{n_2} \\ &\times (A_+)^{n_3} (B_+)^{n_4} \left\langle \frac{n_1+n_2}{2}, -\frac{n_1-n_2}{2} \middle| \frac{n_1}{2}, -\frac{n_1}{2}; \frac{n_2}{2}, \frac{n_2}{2} \right\rangle \\ &\times \left\langle \frac{n_3}{2}, -\frac{n_3}{2}; \frac{n_4}{2}, \frac{n_4}{2} \middle| \frac{n_3+n_4}{2}; -\frac{n_3-n_4}{2} \right\rangle \frac{1}{32\pi^2} \int d\alpha\, d(\cos\beta)\, d\gamma \\ &\times D^{*(n_3+n_4)/2}_{-(n_3+n_4)/2,-(n_3-n_4)/2}(\alpha,\beta,\gamma)\, D^{(n_1+n_2)/2}_{-(n_1+n_2)/2,-(n_1-n_2)/2}(\alpha,\beta,\gamma). \quad (16) \end{aligned}$$

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



We perform the $\alpha, \beta$ and $\gamma$ integration. The result is

$$\left\langle \xi, \xi^{\dagger} \middle| \alpha \right\rangle_{phys} = e^{-|\xi|^2 - \alpha^{\dagger}\alpha} \sum_{n_1, n_2 = 0}^{\infty} |u|^{2(n_1+n_2)} \frac{(A_- A_+)^{n_1} (B_- B_+)^{n_2}}{(n_1! n_2!)^2} \times \left| \left\langle \frac{n_1+n_2}{2}, -\frac{n_1-n_2}{2} \middle| \frac{n_1}{2}, -\frac{n_1}{2}; \frac{n_2}{2}, \frac{n_2}{2} \right\rangle \right|^2, \qquad (17)$$

where $A_- A_+$ and $B_- B_+$ are linear functions of $D^1_{mm'}(\phi, \theta, \psi)$ and $\cos\Theta$. Then the final expression of the coherent states is a function of $D^j_{mm'}(\phi, \theta, \psi)$, $r^{(n_1+n_2)}$ and $\cos\Theta$.

## 4 Conclusion

We have used the mapping between the five-dimensional hydrogen atom and eight oscillators and quantized the latter via path integration over holomorphic coordinates. The kernel governs the parametric time evolution of the eigenstates of the lowering operators and corresponds to the coherent states for the harmonic oscillators. The contribution of the extra degrees of freedom is eliminated by integrating over their final points. We expressed the state functions of the oscillators in terms of the spherical harmonics $D^{1/2}_{mm'}(\phi, \theta, \psi)$ and $D^{1/2}_{nn'}(\alpha, \beta, \gamma)$ and performed the integrals over $\alpha, \beta$ and $\gamma$ by using the orthogonality properties of the $D$ functions. The final expression of the coherent states depends on five physical coordinates of the hydrogen atom.

The Kustaanheimo-Stiefel transformation can be written as $x = u^2$ between the 3-component $x$ and 4-component $u$ quaternions. One of the open problems is to write the Hurwitz transformation in the same way as the Kustaanheimo-Stiefel transformation in terms of the octonions without using the additional degrees of freedom. We used the path integration over the $c$-number holomorphic coordinates $a$ and $a^{\dagger}$, instead of $x$ and $p$. Another open problem is to define the path integrals over the quaternions or octonions.

## Acknowledgments

This work is partly supported by the Akdeniz University (99.01.0105.03).

## References

[1] E. Schrödinger, *Naturwissenschaften* **14**, 664 (1926).

*Fluctuating Paths and Fields*, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).



[2] A. ten Wolde, L.D. Noordam, A. Lagendijk, and H.B. van Linden van den Heuwell, *Phys. Rev. Lett.* **61**, 2099 (1988).

[3] M.M. Nieto and L.M. Simmons Jr., *Phys. Rev. Lett.* **41**, 207 (1978).

[4] L.S. Brown, *Am. J. Phys.* **41**, 525 (1973).

[5] D. Bhaumik, B. Dutta-Roy, and G. Ghosh, *J. Phys: Math. Gen.* **19**, 1355 (1986).

[6] C.C. Gerry, *Phys. Rev. A* **33**, 6 (1988).

[7] T. Toyoda and S. Wakayama, *Phys. Rev. A* **59**, 1021 (1999).

[8] I.H. Duru and H. Kleinert, *Phys. Lett. B* **84**, 185 (1979) and *Fortschr. Phys.* **30**, 401 (1982).

[9] P. Kustaanheimo and E. Stiefel, *J. Reine Angew. Math.* **218**, 204 (1965).

[10] N. Ünal, preprint.

[11] T. Levi-Cività, *Opere Mathematische* **2**, 411 (1956).

[12] F.H.J. Cornish, *J. Phys A: Math. Gen.* **17**, 323 (1984).

[13] A. Hurwitz, *Mathematische Werke*, Band II (Birkhäuser, Basel, 1933), p. 641.

[14] N. Ünal, *Found. Phys.* **28**, 755 (1998).