Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

CRITICAL EXPONENT α OF SUPERFLUID HELIUM FROM VARIATIONAL STRONG-COUPLING THEORY

B. VAN DEN BOSSCHE

Physique Nucléaire Théorique, B5, Université de Liège, 4000 Liège, Belgium,
and
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
E-mail: bossche@physik.fu-berlin.de

The variational strong-coupling theory of Hagen Kleinert is used to determine the critical exponent α of superfluid helium. It is shown that applying the theory to $\exp(\alpha)$, a highly accurate value of α is obtained.

1 Introduction

The variational approach of Feynman and Kleinert [1] has been systematically improved by Kleinert in Ref. [2]. It has been extended to field theory for the determination of critical exponents in $D = 3$ dimensions in Ref. [3] and in $4 - \epsilon$ dimensions in Refs. [4,5]. For a review, see Ref. [6]. Recently, it has been shown that the theory is applicable for the determination of amplitude functions and ratios [7].

Having learned variational perturbation theory directly from its inventor, it is a pleasure to dedicate a contribution on this subject to him. More precisely, I shall focus on the theoretical determination of the critical exponent α of superfluid helium, and show that a negative α is obtained, already at the two-loop level. Our choice of studying this comes from the fact it is probably the best well-known measured quantity. It was obtained in a zero-gravity experiment by Lipa *et al.* [8], who parameterized the specific heat as follows

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

(we use the second of the references quoted in Ref. [8]):

$$C^{\pm} = \frac{A^{\pm}}{\alpha}|t|^{-\alpha}(1 + D|t|^{\Delta} + E|t|^{2\Delta}) + B, \qquad t = T/T_c - 1, \tag{1}$$

$$\alpha = -0.01056 \pm 0.0004, \tag{2}$$

with $\Delta = 0.5, A^+/A^- = 1.0442{\pm}0.001, A^-/\alpha = -525.03, D = -0.00687, E = 0.2152$ and $B = 538.55$ (J/mol K). Apart from B, this parameterization is an approximation to the Wegner expansion form

$$\begin{aligned} F = F_{\pm}|t|^{\chi}\Big(1 &+ a_{0,1}|t|^{\Delta_0} + a_{0,2}|t|^{2\Delta_0} + a_{0,3}|t|^{3\Delta_0} + \cdots \\ &+ a_{1,1}|t|^{\Delta_1} + a_{1,2}|t|^{2\Delta_1} + a_{1,3}|t|^{3\Delta_1} + \cdots\Big) \end{aligned} \tag{3}$$

with χ a combination of critical exponents and $F_{\pm}$ denoting the leading amplitude above and below T_c, respectively. Compared to this general Wegner expansion, higher powers in $\Delta \equiv \Delta_0$ have been neglected in (1), as well as daughter powers $\Delta_i, i \geq 1$. This will be also the case in the present work, where we shall take into account only one exponent Δ, related to the more well-known critical exponent ω by the relation $\Delta = \omega\nu$.

2 Model and Algorithm

The critical behavior of many different physical systems can be described by an $O(N)$-symmetric ϕ^4-theory. In particular, the case $N = 0$ describes polymers, $N = 1$ the Ising transition (a universality class which comprises binary fluids, liquid-vapor transitions and antiferromagnets), $N = 2$ the superfluid transition in helium, $N = 3$ isotropic magnets (transition of the Heisenberg type), and $N = 4$ phase transition of Higgs fields at finite temperature. The field energy is given by the Ginzburg-Landau functional

$$\mathcal{H} = \int d^D x \left[\frac{1}{2}(\nabla\phi_B)^2 + \frac{1}{2}m_B^2\phi_B^2 + \frac{g_B}{4!}(\phi_B^2)^2\right]. \tag{4}$$

As mentioned above, we shall restrict our analysis to the case of superfluid helium, for which the field ϕ_B has $N = 2$ components. The subscript B stands for bare. We shall work in the minimal subtraction (MS) scheme in $4 - \epsilon$ dimensions with ϵ-expansion. For this reason, the square of the bare mass m_B^2 goes to zero at the transition (the critical value of the bare mass is identically zero), linearly with the temperature: $m_B^2 \equiv t = (T/T_c - 1)$, hence the name "reduced temperature" for t.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

To save space, we shall not reproduce the whole algorithm of variational perturbation theory reviewed in Refs. [6,7]. We sketch below only the main points. Let us start with a function f whose expansion in terms of the bare coupling constant is known up to a given order L:

$$f \approx f_L(\bar{g}_B) = \sum_{i=0}^{L} f_i \bar{g}_B^i, \tag{5}$$

with $\bar{g}_B$ a reduced coupling constant to be defined later. Then, provided we know that the function f goes to a constant as the bare coupling constant goes to infinity, variational perturbation theory indicates that the value of the function f at the critical point is given by

$$f_L(\bar{g}_B \to \infty) = \mathrm{opt}_{\hat{g}_B} \left[\sum_{i=0}^{L} f_i \hat{g}_B^i \sum_{j=0}^{L-i} \binom{-i\epsilon/\omega}{j} (-1)^j \right]. \tag{6}$$

This simple formula replaces the well-known but rather involved resummation procedure for the renormalized power series. In particular, for critical exponents, the ϵ-expansion of Eq. (6) reproduces the ϵ-expansion obtained using the renormalized theory (before resummation).

The operator $\mathrm{opt}_{\hat{g}_B}$ is referred to as "optimization", and has a particular meaning: The even-loop orders are optimized by extrema (they may be minima or maxima depending on the critical exponents to be investigated), the odd-loop orders by turning-points, obtained by equating the second derivative of (6) to zero.

We are now ready to determine the exponent α for superfluid helium. We proceed in two steps. The first step (Section 3) is dedicated to two- and three-loop orders, where we can calculate the strong-coupling limit $\lim_{g_B \to \infty} \alpha(g_B)$ analytically. The second step is given in Section 4, where we shall present numerical results up to five loops.

The starting point of our analysis are the five-loop calculations of Refs. [9,10]. Working within the MS scheme means that only the poles in ϵ are removed. This allows us to identify the renormalization constants Z_{m^2}, Z_g, and Z_ϕ which relate the bare and renormalized mass, coupling constant, and field, respectively:

$$m_B^2 = m^2 \frac{Z_{m^2}}{Z_\phi}, \tag{7}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

$$g_B = \mu^\epsilon \frac{Z_g}{Z_\phi^2} g, \tag{8}$$

$$\phi_B = Z_\phi^{1/2} \phi. \tag{9}$$

In the MS scheme, the renormalization constants depend only on g, or, upon inverting (8), on $\bar{g}_B \equiv g_B/|m|$. This statement comes from the identification of the scale μ with the renormalized mass $|m|$ and setting $D = 3$, or $\epsilon = 1$. At the critical point, the renormalized mass goes to zero: The critical theory corresponds to the strong-coupling limit of the bare theory.

From these considerations and the relevant expansions for the $O(2)$-symmetric theory taken from Ref. [6], we obtain

$$\begin{aligned} g = {} & \bar{g}_B - \frac{10}{3}\bar{g}_B^2 + \frac{130}{9}\bar{g}_B^3 - \frac{[6017 + 384\zeta(3)]}{81}\bar{g}_B^4 \\ & + \frac{[420505 + 78432\zeta(3) - 5760\zeta(4) + 36480\zeta(5)]}{972}\bar{g}_B^5 \\ & - \big[26929681 + 9514768\zeta(3) + 92928\zeta(3)^2 - 1260960\zeta(4) \\ & + 8001280\zeta(5) - 912000\zeta(6) + 3386880\zeta(7)\big]\frac{\bar{g}_B^6}{9722}, \end{aligned} \tag{10}$$

$$\begin{aligned} \alpha = {} & \frac{1}{2} - \bar{g}_B + \frac{7}{2}\bar{g}_B^2 - \frac{677}{36}\bar{g}_B^3 + \frac{[81913 + 4272\zeta(3) + 2304\zeta(4)]}{648}\bar{g}_B^4 \\ & - \big[311381 + 46896\zeta(3) - 3520\zeta(3)^2 + 18492\zeta(4) \\ & + 12480\zeta(5) + 15200\zeta(6)\big]\frac{\bar{g}_B^5}{324}, \end{aligned} \tag{11}$$

where we have redefined g and $\bar{g}_B$ so as to absorb a factor $1/(4\pi)^2$. To save space, we have also written these expressions only for $\epsilon = 1$. We have however checked that, by keeping ϵ everywhere, the usual ϵ-expansion of the critical exponents would have been recovered. Taking the exponential of (11), and reexpanding up to the fifth order, we have

$$\begin{aligned} \frac{\exp(\alpha)}{\sqrt{e}} = {} & 1 - \bar{g}_B + 4\bar{g}_B^2 - \frac{809}{36}\bar{g}_B^3 + \frac{[99229 + 4272\zeta(3) + 2304\zeta(4)]}{648}\bar{g}_B^4 \\ & - \big[3788857 + 490320\zeta(3) - 35200\zeta(3)^2 + 196440\zeta(4) \\ & + 124800\zeta(5) + 152000\zeta(6)\big]\frac{\bar{g}_B^5}{3240}. \end{aligned} \tag{12}$$

The aim of this article is to show that the strong-coupling limit of the critical exponent α from the series (12) yields a much better result than the original series (11).

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

3 Analytical Evaluation of α up to Three Loops

Following the general procedure described in Refs. [3–7], the critical exponent ω of the approach to scaling can be extracted from (10) by considering its logarithmic derivative. The result has been obtained analytically at the two-loop level in Ref. [4] and at the three-loop level in Ref. [7]. Specializing to the case of superfluid helium, the solutions read, with $\omega = 1/(\rho - 1)$, and where the subscript indicates the loop order,

$$\rho_2 = 4\sqrt{\frac{2}{5}}, \tag{13}$$

$$\rho_3 = \frac{50311 + 1152\zeta(3)}{2\,[12907 - 3456\zeta(3)]} - 3\frac{\sqrt{3\,[1039 + 128\zeta(3)]\,[62779 + 2688\zeta(3)]}}{[12907 - 3456\zeta(3)]} \times \cos\left[-\frac{2}{3}\pi + \frac{1}{3}\mathrm{arcsec}(\theta)\right], \tag{14}$$

with

$$\theta = \frac{400\,[62779 + 2688\zeta(3)]^{3/2}}{\sqrt{3117 + 384\zeta(3)}\,\{55818649 + 768\zeta(3)\,[-118163 + 15552\zeta(3)]\}}. \tag{15}$$

Numerically, we have $\rho_2 \approx 2.52982$ and $\rho_3 \approx 2.38683$, to which correspond $\omega_2 \approx 0.65367$ and $\omega_3 \approx 0.721069$.

Using (6) to two loops, together with (13), the strong-coupling limit of (11) and (12) is

$$\alpha_2 = \frac{3}{70} \approx 0.0428571, \tag{16}$$

$$\alpha_2 = \frac{1}{2} - \ln\left(\frac{5}{3}\right) \approx -0.0108256, \tag{17}$$

respectively, from which it is clear that the strong-coupling limit of the direct series (11) fails to give a negative value of α at the two-loop level, while the strong-coupling limit of $\exp(\alpha)$ leads to such a negative value. Moreover, this value is extremely close to the experimental result (2).

The three-loop order is also determined analytically and confirms the two-loop conclusion: Using (6) to three loops, together with (14) and (15), the strong-coupling limit of (11) and (12) is

$$\alpha_3 = \frac{450097 + 77826\rho_3 - 127218\rho_3^2 + 8988\rho_3^3}{916658} \approx 0.0363442, \tag{18}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Table 1. Critical exponent ω of the approach to scaling for different loop orders L.

L	2	3	4	5
ρ_L	2.52982	2.38683	2.36773	2.32803
ω_L	0.653671	0.721069	0.731141	0.752997

Table 2. Critical exponent α for different loop orders L.

L	2	3	4	5
α_L from Eq. (11)	0.0428571	0.0363441	0.0214116	0.0189176
α_L from Eq. (12)	−0.0108256	−0.0116665	−0.0110531	−0.0116324

$$\begin{aligned}\alpha_3 &= \frac{1}{2} + \ln\left(\frac{648337 + 56280\rho_3 - 93144\rho_3^2 + 10320\rho_3^3}{654481}\right) \\ &\approx -0.0116664,\end{aligned} \tag{19}$$

respectively. The second result lies again very close to the experimental value (2), whereas the direct evaluation would have led to a positive value. The excellent agreement of the strong-coupling limit of (12) is confirmed by the numerical five-loop calculation of the next section.

4 Numerical Evaluation of α to Five Loops

It is not possible to evaluate analytically the strong-coupling limit of a given function above the three-loop order since the defining equation for ρ, obtained taking the logarithmic derivative of (10) and applying the algorithm (6), is higher than cubic. Nothing can, however, prevent us from calculating the algorithm numerically. We have done it for the four- and five-loop orders. To facilitate the comparison with the previous section, we recall also below the numerical value of the second- and third-loop order.

With $N = 2$ and $\epsilon = 1$, we obtain for ρ_L and ω_L the values shown in Table 1. Applying (6) to (11) and (12), together with the numerical values of Table 1, we obtain for the critical exponent α results which are summarized in Table 2. We see that the strong-coupling limit of α, as given by the direct series (11), decreases, as expected, but is still positive at the five-loop order. A

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann
(World Scientific, Singapore, 2001).

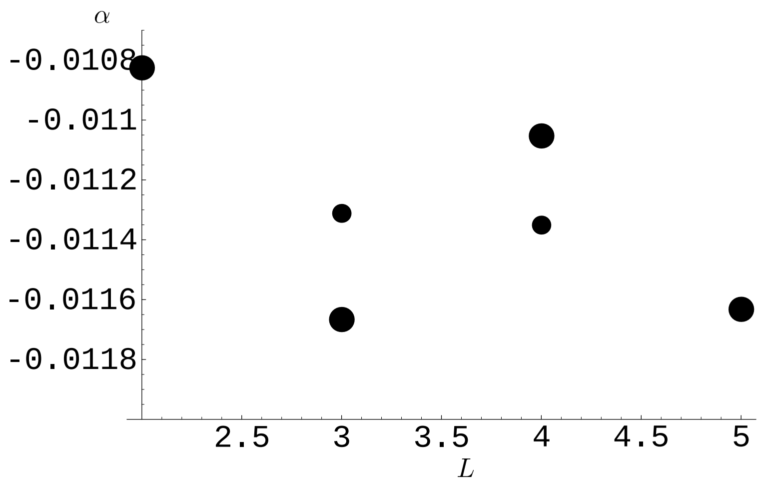

Figure 1. Critical exponent α of superfluid helium as a function of the loop order L. Thick dots denote the exponent based on the strong-coupling limit of Eq. (12) while thin dots are Shanks-improved results [12].

nontrivial extrapolation procedure, such as the one employed in Refs. [3,5,6], is needed.

The calculation of α, based on (12), gives negative results, not too far from the experimental result. Moreover, the results seem to alternate around a given value, see Fig. 1. An ordinary procedure for accelerating the convergence seems to be applicable. In our case, it is tempting to perform a Shanks transformation [12]. Denoting by α^* the improved value, we can construct

$$\alpha_3^* = \frac{\alpha_2\alpha_4 - \alpha_3^2}{\alpha_2 + \alpha_4 - 2\alpha_3} = -0.011312, \tag{20}$$

$$\alpha_4^* = \frac{\alpha_3\alpha_5 - \alpha_4^2}{\alpha_3 + \alpha_5 - 2\alpha_4} = -0.011351. \tag{21}$$

In Figure 1, we have plotted the values given in the last raw of Table 2 as well as the improved values of Eqs. (20) and (21).

It would be extremely interesting to obtain the six-loop order of α. Then a third Shanks-improved point α_5^* could be obtained, and an iterated Shanks transformation could be performed, allowing to obtain $\alpha^{**} = [\alpha_3^*\alpha_5^* - (\alpha_4^*)^2]/(\alpha_3^* + \alpha_5^* - 2\alpha_4^*)$. This would probably lead to an extremely precise value of α. Due to the large number of Feynman diagrams to be evaluated, this task is, however, not manageable at present using available

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

techniques, and calls for new ideas.

Another interesting work would be to check if the Borel resummation of the ϵ-expansion of $\exp(\alpha)$ would lead to a similar improvement of the convergence as variational perturbation theory.

5 Conclusion

In this contribution, we have applied variational perturbation theory to the determination of the critical exponent α of superfluid helium from five-loop ϵ-expansions. Previous studies were based on Borel resummation [11], or on variational perturbation theory together with a suitable extrapolation procedure [3,5,6]. The approach followed here was based on a full self-consistent calculation, using the same loop order for the critical exponent of the approach to scaling ω as for the exponent α itself. For the direct resummation of α, this has proven to be of less accuracy than the extrapolation approach [5,6]. However, we have shown here that an appropriate choice of the function of α, in this work we have chosen $\exp[\alpha(\bar{g}_B)]$, to be evaluated in the strong-coupling limit $\bar{g}_B$ is able to drastically improve the convergence. Our work does not solve the question of which function of a critical exponent gives the fastest convergence of strong-coupling theory, nor have we checked whether the exponential function can be used to obtain the other exponents more accurately than before. The present note may, however, be seen as a first step of a program towards optimizing the functions of the critical exponents to be evaluated.

Acknowledgments

I thank the Alexander von Humboldt Foundation and the F.N.R.S. (Belgium) for support. I am indebted to Professor Kleinert for numerous clarifications and discussions.

References

[1] R.P. Feynman and H. Kleinert, *Phys. Rev. A* **34**, 5080 (1986).

[2] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995); variational perturbation theory is developed in Chapters 5 and 17.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

[3] H. Kleinert, *Phys. Rev. D* **57**, 2264 (1998); Addendum *ibid.* **58**, 107702 (1998); *ibid.* **60**, 085001 (1999).

[4] H. Kleinert, *Phys. Lett. B* **434**, 74 (1998); **463**, 69 (1999).

[5] H. Kleinert and V. Schulte-Frohlinde, *J. Phys. A* **34**, 1037 (2001), eprint: cond-mat/9907214.

[6] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4-Theories* (World Scientific, Singapore, 2001) [http://www.physik.fu-berlin.de/~kleinert/b8].

[7] H. Kleinert and B. Van den Bossche, eprint: cond-mat/0011329.

[8] J.A. Lipa, D.R. Swanson, J.A. Nissen, T.C.P. Chui, and U.E. Israelsson, *Phys. Rev. Lett.* **76**, 944 (1996); J.A. Lipa, D.R. Swanson, J.A. Nissen, Z.K. Geng, P.R. Williamson, D.A. Stricker, T.C.P. Chui, U.E. Israelsson, and M. Larson, *Phys. Rev. Lett.* **84**, 4894 (2000). In the second paper, the value of α given in the first paper was corrected to $\alpha = -0.01056 \pm 0.0004$.

[9] K.G. Chetyrkin, S.G. Gorishny, S.A. Larin, and F.V. Tkachov, *Phys. Lett. B* **132**, 351 (1983).

[10] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S.A. Larin, *Phys. Lett. B* **272**, 39 (1991); Erratum: *Phys. Lett. B* **319**, 545(E) (1993).

[11] R. Guida and J. Zinn-Justin, *J. Phys. A* **31**, 8103 (1998), eprint: cond-mat/9803240 (1998). See also J. Zinn-Justin, eprint: hep-th/0002136.

[12] C. Brezinski and M. Redivo-Zaglia, *Extrapolation Methods, Theory, and Practice*, in *Studies in Computational Mathematics*, Eds. C. Brezinski and L. Wuytack (North-Holland, Amsterdam, 1991).