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TOPOLOGICAL SINGULARITIES, DEFECT FORMATION, AND PHASE TRANSITIONS IN QUANTUM FIELD THEORY

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By resorting to results in quantum field theory some light is shed on the mechanism of the formation of topological defects in the process of phase transitions.

1 Symmetry Breaking Phase Transitions and Topological Singularities

I am pleased to dedicate this article to Hagen Kleinert on the occasion of his sixtieth birthday. I met Hagen for the first time about 25 years ago, at one of the Karpacz Winter Schools in Theoretical Physics in Poland. Since then a sincere friendship has been established. Besides the reciprocal sympathy, such a friendship finds perhaps its roots in the sense of humor with which both of us look at life and theoretical physics.

In this article I discuss, from the standpoint of quantum field theory (QFT), why topological defects, such as vortices, are formed in the process of symmetry breaking phase transitions and which effects boundaries and temperature have on defect formation [1,2]. Topological defects are described in QFT as extended objects created by the non-homogeneous condensation of Nambu-Goldstone (NG) modes carrying a topological singularity [3,4]. Therefore my attention is focused on the non-homogeneous boson condensation. Topological defects appear in many systems in a wide range of energy scales [5,6], from condensed matter physics to cosmology. I hope that from my discussion will emerge the unified view of collective phenomena in

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many physically different systems which Hagen Kleinert has pursued during the many years of his dense research activity. The results have been obtained by the use of tools such as group theory and path integrals which have been and are Hagen's preferred fields of research.

I begin by recalling the mechanism of dynamical rearrangement of symmetry by which boson field translations are introduced in the theory. Then I discuss how macroscopic fields and currents are generated by boson condensation. By using such results, I will show that phase transitions in a gauge theory involve non-homogeneous boson condensation with a topological singularity and hence that topological defects appear in the process of symmetry breaking phase transitions. Finally, I will discuss finite temperature and finite volume effects.

To be specific I start by considering a complex scalar Heisenberg field $\phi_H(x)$ interacting with a gauge field $A_{H,\mu}(x)$ [7,8]. The Lagrangian density $\mathcal{L}[\phi_H(x), {\phi_H}^*(x), A_{H,\mu}(x)]$ is assumed to be invariant under global and local gauge transformations:

$$\phi_H(x) \to e^{i\theta}\phi_H(x) \ , \qquad A_{H,\mu}(x) \to A_{H,\mu}(x) \ , \tag{1}$$
$$\phi_H(x) \to e^{ie_0\lambda(x)}\phi_H(x) \ , \qquad A_{H,\mu(x)} \to A_{H,\mu}(x) \, + \, \partial_\mu\lambda(x) \ , \tag{2}$$

respectively. I assume $\lambda(x) \to 0$ for $|x_0| \to \infty$ and/or $|\mathbf{x}| \to \infty$. I use the Lorentz gauge $\partial^\mu A_{H,\mu}(x) = 0$ and set $\phi_H(x) = [\psi_H(x) + i\chi_H(x)]/\sqrt{2}$. I also assume that spontaneous breakdown of symmetry (SBS) can occur: $\langle 0|\phi_H(x)|0\rangle \equiv \tilde{v} \neq 0$, with constant $\tilde{v}$ and with $\rho_H(x) \equiv \psi_H(x) - \tilde{v}$. The generating functional is [9]

$$W[J,K] = \frac{1}{N}\int [dA_\mu][d\phi][d\phi^*][dB] \, \exp\left[i\int d^4x\,(\mathcal{L}(x) + B(x)\partial^\mu A_\mu(x) \right.$$
$$\left. + K^*\phi + K\phi^* + J^\mu(x)A_\mu(x) + i\epsilon|\phi(x) - \tilde{v}|^2) \right] \ , \tag{3}$$

where N is a convenient normalization. In the gauge constraint term, $B(x)$ is an auxiliary field. The ϵ-term specifies the condition of breakdown of symmetry under which we want to compute the path integral. It may represent the small external field triggering the symmetry breakdown. The limit $\epsilon \to 0$ must be performed at the end of the computations.

The LSZ maps (dynamical maps) between Heisenberg field and asymptotic

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(also called physical) in- (or out-) fields are:

$$\phi_H(x) = :\exp\left\{ i\frac{Z_\chi^{\frac{1}{2}}}{\tilde{v}}\chi_{\rm in}(x)\right\}\left[\tilde{v} + Z_\rho^{\frac{1}{2}}\rho_{\rm in}(x) + F\right]:, \qquad (4)$$

$$A_H^\mu(x) = Z_3^{\frac{1}{2}}U_{\rm in}^\mu(x) + \frac{Z_\chi^{\frac{1}{2}}}{e_0\tilde{v}}\partial^\mu b_{\rm in}(x) + :F^\mu: \; . \qquad (5)$$

The functionals $F \equiv F[\rho_\in, U_{\rm in}^\mu, \partial(\chi_{\rm in} - b_{\rm in})]$ and $F^\mu \equiv F^\mu[\rho_{\rm in}, U_{\rm in}^\mu, \partial(\chi_{\rm in} - b_{\rm in})]$ are determined within a particular model. The S-matrix is given by $S =: S[\rho_{\rm in}, U_{\rm in}^\mu, \partial(\chi_{\rm in} - b_{\rm in})] :$, and I use $A_H^{0\mu}(x) \equiv A_H^\mu(x) - e_0\tilde{v} : \partial^\mu b_{\rm in}(x) :$. The field $\chi_{\rm in}$ denotes the NG mode, $b_{\rm in}$ the ghost mode, $\rho_{\rm in}$ the massive matter field, and $U_{\rm in}^\mu$ the massive vector field. Their respective equations are

$$\partial^2\chi_{\rm in}(x) = 0\,, \quad \partial^2 b_{\rm in}(x) = 0\,, \quad (\partial^2 + m_\rho^2)\rho_{\rm in}(x) = 0\,, \qquad (6)$$

$$(\partial^2 + {m_V}^2)U_{\mu{\rm in}}(x) = 0\,, \qquad \partial^\mu U_{\mu{\rm in}}(x) = 0\,, \qquad (7)$$

with ${m_V}^2 = Z_3(e_0\tilde{v})^2/Z_\chi$. One also has

$$B_H(x) = \frac{e_0\tilde{v}}{Z_\chi^{\frac{1}{2}}}[b_{\rm in}(x) - \chi_{\rm in}(x)]\;, \qquad (8)$$

$$\partial^2 B_H(x) = 0\,, \quad -\partial^2 A_{H\mu}(x) = j_{H\mu}(x) - \partial_\mu B_H(x)\,, \qquad (9)$$

with $j_{H\mu}(x) = \delta\mathcal{L}(x)/\delta A_H^\mu(x)$. If one requires that the current $j_{H\mu}$ is the only source of the gauge field $A_{H\mu}$ in any observable process, one has to impose the condition ${}_p\langle b|\partial_\mu B_H(x)|a\rangle_p = 0$, i.e.

$$(-\partial^2)\,{}_p\langle b|{A^0}_{H\mu}(x)|a\rangle_p = {}_p\langle b|j_{H\mu}(x)|a\rangle_p\;. \qquad (10)$$

Here $|a\rangle_p$ and $|b\rangle_p$ denote two generic physical states. The equations (10) are the classical Maxwell equations. The condition ${}_p\langle b|\partial_\mu B_H(x)|a\rangle_p = 0$ leads to the Gupta-Bleuler-like condition

$$[\chi_{\rm in}^{(-)}(x) - b_{\rm in}^{(-)}(x)]|a\rangle_p = 0\,, \qquad (11)$$

where $\chi_{\rm in}^{(-)}$ and $b_{\rm in}^{(-)}$ are the positive-frequency parts of the corresponding fields $\chi_{\rm in}$ and $b_{\rm in}$ which do not participate in any observable reaction. However, I stress that NG bosons do not disappear from the theory. As we will see, their condensation in the vacuum can have observable effects.

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Finally, one finds that the $U(1)$ local and global gauge transformations [Eqs. (2) and (1)] of the Heisenberg fields are induced by the in-field transformations

$$\chi_{\rm in}(x) \to \chi_{\rm in}(x) + \frac{e_0 \tilde{v}}{Z_\chi^{\frac{1}{2}}} \lambda(x) \;, \quad b_{\rm in}(x) \to b_{\rm in}(x) + \frac{e_0 \tilde{v}}{Z_\chi^{\frac{1}{2}}} \lambda(x) \;, \tag{12}$$

$$\rho_{\rm in}(x) \to \rho_{\rm in}(x) \;, \quad U^\mu_{\rm in}(x) \to U^\mu_{\rm in}(x) \;, \tag{13}$$

and by

$$\chi_{\rm in}(x) \to \chi_{\rm in}(x) + \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}} \theta f(x) \;, \tag{14}$$

$$b_{\rm in}(x) \to b_{\rm in}(x) \;, \quad \rho_{\rm in}(x) \to \rho_{\rm in}(x) \;, \quad U^\mu_{\rm in}(x) \to U^\mu_{\rm in}(x) \;, \tag{15}$$

with $\partial^2 f(x) = 0$, respectively. Eq. (14) with $f(x) = 1$, which describes the homogeneous boson condensation, is not unitarily implementable. It induces transitions among unitarily inequivalent Fock spaces. The function $f(x)$ makes the generator of such a transformation well defined. The limit $f(x) \to 1$ is to be performed at the end of the computation. The fact that the Heisenberg field transformations are induced by Eqs. (12)-(15) is named the *dynamical rearrangement of symmetry* [3,4]. The in-field equations and the S-matrix are invariant under the above in-field transformations, and B_H is changed by an irrelevant c-number under (14) and (15) (in the limit $f \to 1$). It can be shown that the group of the transformations under which the in-field equations are invariant is the group contraction of the symmetry group for the Heisenberg field equations [10].

1.1 How Macroscopic Field and Current are Generated by Boson Condensation

Translations of bosonic physical fields (not necessarily massless) by space-time dependent functions, say $\alpha(x)$, satisfying the same field equation of the translated physical field, are called *boson transformations* [3]. Eqs. (12) and (14) (with $\partial^2\lambda(x) = 0$ and $\partial^2 f(x) = 0$) are examples of boson transformations. Consider the transformation $\phi_H[x; \chi_{\rm in}(x)] \to \phi'_H \equiv \phi_H[x; \chi_{\rm in}(x) + \alpha(x)]$. The *boson transformation theorem* states that ϕ'_H is also a solution of the Heisenberg field equation for ϕ_H [3,4].

It can be shown that, in the presence of a gauge field, the boson transformation with regular (i.e. Fourier transformable) $\alpha(x)$ is equivalent to a gauge transformation. The only effect is the appearance of a phase factor in the

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order parameter: $\tilde{v}(x) = e^{ic\alpha(x)}\tilde{v}$, with a constant c. Observable quantities are thus not affected in such a case. Note that in a theory which has only global gauge invariance non-singular boson transformations of the NG fields can produce non-trivial physical effects (like linear flow in superfluidity).

The proof of the boson transformation theorem relies on the fact that $\alpha(x)$ is a regular function. If $\alpha(x)$ carries some singularity (divergence or topological singularity) the singular region must be excluded when integrating on space and/or time. For example, if $\alpha(x)$ is singular on the axis of a cylinder (at $r = 0$), the singular line $r = 0$ must be excluded by a cylindrical surface of infinitesimal radius. The phase of the order parameter will be singular on that line. This means that SBS does not occur in that region (the core): there one has the "normal" state rather than the ordered one.

Since translation of a boson field describes boson condensation, we see that boson transformations describe *non-homogeneous* boson condensation. The boson theorem then shows that the same dynamics (the same field equations) may describe homogeneous and non-homogeneous phenomena. This leads us directly to the mechanism of formation of extended objects, which are in fact created by the non-homogeneous boson condensation [3,4,11]. Also, since different phases (described in QFT by unitarily inequivalent representations) are associated to different NG boson condensation densities, we see that, by inducing variations of NG boson condensation, boson transformations represent transitions through physically different phases of the system. This establishes a connection between the formation of extended objects and the process of phase transitions.

Boson transformations must be also compatible with the physical state condition (11). Under the boson transformation $\chi_{\rm in}(x) \to \chi_{\rm in}(x) + (\tilde{v}/Z_\chi^{\frac{1}{2}})f(x)$, B_H changes as

$$B_H(x) \to B_H(x) - \frac{e_0\tilde{v}^2}{Z_\chi} f(x) \ . \tag{16}$$

Eq. (10) is violated upon imposing the Gupta-Bleuler-like condition. In order to restore it, one must compensate the shift in B_H by means of the transformation of $U_{\rm in}$:

$$U^\mu_{\rm in}(x) \to U^\mu_{\rm in}(x) + {Z_3}^{-\frac{1}{2}} a^\mu(x) \ , \qquad \partial_\mu a^\mu(x) = 0 \ , \tag{17}$$

with a convenient c-number function $a^\mu(x)$. The dynamical maps of the

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various Heisenberg operators are not affected by (17) provided

$$(\partial^2 + m_V^2)a_\mu(x) \; = \; \frac{m_V^2}{e_0}\partial_\mu f(x) \; . \tag{18}$$

This is the Maxwell equation for the vector potential a_μ [9,12]. The classical ground-state current j_μ is

$$j_\mu(x) \equiv \langle 0|j_{H\mu}(x)|0\rangle \; = \; m_V^2 \left[a_\mu(x) - \frac{1}{e_0}\partial_\mu f(x) \right] \; , \tag{19}$$

where $m_V^2 a_\mu(x)$ is the *Meissner current* and $m_V^2 \partial_\mu f(x)/e_0$ the *boson current*. Note that the classical current is given in terms of variations $\partial_\mu f$ of the non-homogeneous boson condensate density.

Summarizing, the macroscopic field and current are expressed in terms of the boson condensation function.

1.2 Why Do Topological Defects Exist Only in the Presence of Massless Bosons?

Let us now show that boson transformation functions carrying topological singularities are allowed only for massless bosons [3,4].

Consider the boson transformation $\chi_{\rm in}(x) \to \chi_{\rm in}(x) + f(x)$. Let $f(x)$ carry a topological singularity making it path-dependent such that it fails to satisfy the integrability condition of Schwarz:

$$G^\dagger_{\mu\nu}(x) \equiv [\partial_\mu, \partial_\nu] \, f(x) \neq 0 \tag{20}$$

for certain μ, ν, x. We have seen that $\partial_\mu f$ is related to observables and therefore it is single-valued, i.e. $[\partial_\rho, \partial_\nu] \, \partial_\mu f(x) \; = \; 0$. Recall that $f(x)$ is a solution of the $\chi_{\rm in}$ equation, and suppose that $\chi_{\rm in}$ is massive: $(\partial^2 + m^2) f(x) = 0$. It follows from the regularity of $\partial_\mu f(x)$ that

$$\partial_\mu f(x) \; = \; \frac{1}{\partial^2 \, + \, m^2} \partial^\lambda \, G^\dagger_{\lambda\mu}(x) \; , \tag{21}$$

which leads to $\partial^2 f(x) \; = \; 0$, which in turn implies $m = 0$. Thus (20) is compatible only with a *massless* equation for $\chi_{\rm in}$. This explains why topological defects are observed always in the presence of NG bosons, namely of ordering induced by NG condensate. The topological charge is defined as

$$N_T = \int_C dl^\mu \, \partial_\mu \, f = \int_S dS_\mu \epsilon^{\mu\nu\sigma} \, \partial_\nu \partial_\sigma \, f \; = \; \frac{1}{2} \int_S dS^{\mu\nu} \, G^\dagger_{\mu\nu} \; , \tag{22}$$

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where C is a contour enclosing the singularity and S a surface with C as a boundary. The charge N_T does not depend on the path C provided it does not cross the singularity. The tensor $G^{\mu\nu}$ is $G^{\mu\nu}(x) \equiv -\frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}G^{\dagger}_{\lambda\rho}(x)$. It satisfies the continuity equation

$$\partial_\mu\, G^{\mu\nu}(x) \,=\, 0 \qquad \Leftrightarrow \qquad \partial_\mu\, G^{\dagger}_{\lambda\rho} \,+\, \partial_\rho\, G^{\dagger}_{\mu\lambda} \,+\, \partial_\lambda\, G^{\dagger}_{\rho\mu} \,=\, 0\;. \tag{23}$$

This completely characterizes the topological charge of the extended object [4].

On the other hand, the macroscopic ground-state effects do not occur for regular $f(x)$ ($G^{\dagger}_{\mu\nu} = 0$). In fact, from Eq. (18) we obtain $a_\mu(x) = \partial_\mu f(x)/e_0$ for regular f which implies a zero classical current ($j_\mu = 0$) and a zero classical field ($F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$) (the Meissner and the boson current cancel each other).

In conclusion, the vacuum current appears only when $f(x)$ has topological singularities and this is allowed only for the condensation of massless bosons, i.e. when SBS occurs. We thus see that, in a gauge theory, the symmetry breaking phase transitions characterized by macroscopic ground-state effects, such as the vacuum current and field (as in superconductors), can occur only when there are non-zero gradients of topologically non trivial, non-homogeneous condensation of NG bosons. *Since these are also the conditions for the formation of topological defects, we also see why topological defects are observed in the process of phase transitions.*

Notice that the appearance of a space-time order parameter is no guarantee that persistent ground-state currents (and fields) will exist: if f is a regular function, the space-time dependence of $\tilde{v}$ can be gauged away by an appropriate gauge transformation.

Since the boson transformation with regular f does not affect observable quantities, the S matrix is actually given by

$$S \,=: S[\rho_{\rm in}, U^\mu_{\rm in} - \frac{1}{m_V}\partial(\chi_{\rm in} - b_{\rm in})] :\;. \tag{24}$$

This is in fact independent of the boson transformation with regular f:

$$S \,\rightarrow\, S' =: S[\rho_{\rm in}, U^\mu_{\rm in} - \frac{1}{m_V}\partial(\chi_{\rm in} - b_{\rm in}) + Z_3^{-\frac{1}{2}}(a^\mu - \frac{1}{e_0}\partial^\mu f)] :, \tag{25}$$

since $a_\mu(x) = \partial_\mu f(x)/e_0$ for regular f. However, $S' \neq S$ for singular f: S' includes the interaction of the quanta $U^\mu_{\rm in}$ and $\phi_{\rm in}$ with the classical field and current. Thus we see how quantum fluctuations may interact and have effects

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on classically behaving macroscopic defects: *our picture includes interaction of quanta with macroscopic objects.*

2 Temperature and Volume Effects

The condition of breakdown of symmetry at finite temperature in the case of non-homogeneous condensation is [11]

$$\langle 0(\beta)|\phi(x)|0(\beta)\rangle = \frac{1}{\sqrt{2}}\sigma(x,\beta) \ , \tag{26}$$

where $\beta \equiv 1/k_B T$. Here, $|0(\beta)\rangle$ denotes the temperature-dependent vacuum state in Thermo Field Dynamics [3,4,13]. Note that the statistical average of some operator A is given by $\langle A\rangle_0 = \langle 0(\beta)|\phi(x)|0(\beta)\rangle$.

The fields $\phi \equiv \rho + \sigma(x,\beta)/\sqrt{2}$, χ and A_μ may undergo translation transformations by c-number functions, say σ, κ and α_μ, respectively, controlling the respective condensate structures. Usual gauge transformations are induced by using $\sigma = 0$, $\kappa = \alpha(x)$ and $\alpha_\mu(x) = \partial_\mu \alpha(x)$. The parameters m and λ will denote the Higgs field mass and self-coupling, respectively; $\tilde{v} \equiv \langle 0|\phi|0\rangle$ at $T = 0$ is assumed to be non-zero, M is the gauge field mass, e the coupling between A_μ and ϕ.

The vortex solution can be studied by introducing cylindrical coordinates. The asymptotic gauge field configuration is imposed by considering the pure angular function as a gauge function at infinity $\kappa(x) = n\,\theta/e$, where $\kappa(x) = 0$ at $r = 0$:

$$\alpha^i_{as} = -\frac{n}{er}\,\mathbf{e}^i_\theta \ . \tag{27}$$

Here n is the winding number. One can show [11] that the masses are given by

$$m^2(x) = 2\lambda\sigma_0^2 f^2(x) \quad , \quad M^2(x) = e^2(\sigma_0^2 f^2(x) + \langle :\bar{\rho}^2:\rangle_0) \quad , \tag{28}$$

with σ_0 the Higgs field condensate which goes to zero at the critical temperature T_C, and to $\tilde{v}$ at $T = 0$; $\bar{\rho}$ denotes the physical field. These masses act as potential terms in the field equations and only at spatial infinity $(r \to \infty, f(x) \to 1)$ ordinary mass interpretation is recovered. In fact one has the asymptotic behavior

$$K(r) \simeq e^{-Mr} = e^{-r/R_0} \quad , \quad f(r) \simeq 1 - f_0 e^{-mr} = 1 - f_0 e^{-r/r_0} \quad . \tag{29}$$

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$R_0 \equiv 1/M$ gives the size of the gauge field core and $r_0 \equiv 1/m$ the Higgs field core. $K(r)$ is related to the α function. Symmetry is restored above T_C.

In the case of the kink solution, the $\rho^{\rm in}_\beta(x)$ condensation is induced by the boson transformation with $f_\beta(x) = \text{const.} \cdot e^{-\mu_0(\beta)x_1}$ playing the role of "form factor". The number of condensed bosons is proportional to $|f_\beta(x)|^2 = e^{-2\mu_0(\beta)\,(x_1-a)}$, which is maximal near the kink center $x_1 = a$ and decreases over a size $\xi_\beta = 2/\mu_0(\beta)$. The boson translation by f_β breaks the homogeneity of the order parameter $v(\beta)$ which is otherwise constant in space.

The mass $\mu_0 = (2\lambda)^{1/2} v(\beta)$ of the "constituent" fields $\rho^{\rm in}$ fixes the kink size $\xi_\beta \propto 2/\mu_0 = \sqrt{2}/\sqrt{\lambda}v(\beta)$ which thus increases as $T \to T_C$ (say $T \neq T_C$ but near T_C). In the $T \to 0$ limit the kink size is $\xi_0 \propto \sqrt{2}/\sqrt{\lambda}\tilde{v} < \sqrt{2}/\sqrt{\lambda}v(\beta) = \xi_\beta$, since [11] $v^2(\beta) = \tilde{v}^2 - 3\langle: \rho^2 :\rangle_0 < \tilde{v}^2$.

For T different from zero, the thermal Bose condensate $\langle: \rho^2 :\rangle_0$ develops which acts as a potential term for the kink field. Such a potential term controls the "size" and the number of the kinks. Only in the limit $v(x, \beta) \to$ const. the $\rho^{\rm in}_\beta(x)$ field may be considered as a free field, e.g. far from the kink core.

Finally, let me consider finite-volume effects. Suppose we have homogeneous boson condensation. For large but finite volume, one expects that the condition of symmetry breakdown is satisfied "inside the bulk" *far* from the boundaries. However, "near" the boundaries, one might expect "distortions" in the order parameter: $\tilde{v} = \tilde{v}(x)$ (or even $\tilde{v} \to 0$): "near" the system boundaries we may have a non-homogeneous order parameter. Non-homogeneities in the boson condensation will "smooth out" in the $V \to \infty$ limit. Let me put $V \equiv \eta^{-3}$. Then, the NG mode acquires an effective gap (mass) of the order of η ($\omega_\eta^2 = \eta^2$). The Goldstone theorem (existence of gapless modes) is recovered in the infinite volume limit ($\eta \to 0$). The effect of the boundaries ($\eta \neq 0$) is to give an "effective mass" $m_{\rm eff} \equiv \omega_\eta$ to the NG bosons. These will then propagate over a range of the order of $\xi \equiv 1/\eta$, which is the system linear size.

Note that only if $\epsilon \neq 0$ (cf. Eq. (3)), the order parameter can be kept different from zero, i.e. if $\eta \neq 0$, then ϵ must be non-zero in order to have $\tilde{v} \neq 0$. In such a case the symmetry breakdown is maintained thanks to the non-zero ϵ which acts as an external field acting as a pump providing energy which is required in order to condense modes of non-zero lowest energy ω_η. Boundary effects are thus in competition with the breakdown of symmetry [1,2]. They may preclude its occurrence or, if symmetry is already broken, they

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may reduce the order parameter to zero.

Temperature may have similar effects on the order parameter (at T_C symmetry may be restored). Since the order parameter goes to zero in the absence of the external supply of energy, when NG modes acquire non-zero effective mass, one may represent the effect of thermalization in terms of finite-volume effects and get $\eta \propto \sqrt{|T-T_C|/T_C}$. In this way, temperature changes near T_C may be discussed as variations of the condensate domain size ξ. For example, in the presence of an external driving field ($\epsilon \neq 0$), for $T > T_C$ (but near to T_C) one may have the formation of ordered domains of size $\xi \propto (\sqrt{|T-T_C|/T_C})^{-1}$, *before* the transition to the fully ordered phase is achieved (as $T \to T_C$). As far as $\eta \neq 0$, the ordered domains are unstable; they disappear as the external field coupling $\epsilon \to 0$.

Of course, if ordered domains are still present at $T < T_C$, they also disappear as $\epsilon \to 0$. The possibility to maintain such ordered domains below T_C depends on the speed at which T is lowered, compared to the speed at which the system is able to become homogeneously ordered. Notice that the speed of $T \to T_C$ is related to the speed of $\eta \to 0$.

The order parameters $v(x,\beta)$ and $\sigma(x,\beta)$ provide a mapping between the variation domains of (x,β) and the *space of the unitarily inequivalent representations* of the canonical commutation relations, i.e. the set of Hilbert spaces where the operator field ϕ is realized for different values of the order parameter. As is well known, one has the mapping π of S^1 in the vortex case, surrounding the $r=0$ singularity, to the group manifold of $U(1)$ which is topologically characterized by the winding number $n \in Z \in \pi_1(S^1)$. It is such a singularity which is carried by the boson condensation function of the NG modes. In the monopole case [11], the mapping π is the one of the sphere S^2, surrounding the singularity $r=0$, to $SO(3)/SO(2)$ group manifold, with homotopy classes of $\pi_2(S^2) = Z$. Again, the singularity is carried by the NG boson condensation function. The same situation occurs in the sphaleron case [11], provided one replaces $SO(3)$ and $SO(2)$ with $SU(2)$ and $U(1)$, respectively.

To conclude let me state that phase transitions imply "moving" over unitarily inequivalent representations, and this implies in general a non-trivial homotopy mapping between the (x,β) variability domain and the group manifold. The invariance of the theory under the involved symmetry group then leads to NG boson condensation with topological singularities. The conditions for the formation of topological defects are then satisfied. *This explains why we observe topological defect formation in the process of phase transitions.*

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In the case of the kink there are no NG modes. Nevertheless the topologically non-trivial kink solution requires the boson condensation function to carry divergence singularity at spatial infinity.

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