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Functional Closure of Schwinger–Dyson Equations in Quantum Electrodynamics

1. Generation of Connected and One-Particle Irreducible Feynman Diagrams

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Using functional derivatives with respect to free propagators and interactions we derive a closed set of Schwinger–Dyson equations in quantum electrodynamics. Its conversion to graphical recursion relations allows us to systematically generate all connected and one-particle irreducible Feynman diagrams for the n-point functions and the vacuum energy together with their correct weights.

I. INTRODUCTION

In quantum field theory, the calculation of physical quantities usually relies on evaluating Feynman integrals which are pictured by diagrams. Each diagram is associated with a certain weight depending on its topology. There exist various convenient computer programs, for instance *FeynArts* [1–3] or *QGRAF* [4, 5], for constructing these diagrams and for determining their weights in different field theories. Some of them are based on a combinatorial enumeration of all possible ways of connecting vertices by lines according to Feynman's rules. Others use a systematic generation of homeomorphically irreducible star graphs [6, 7]. The latter approach is quite efficient and popular at higher orders; it has, however, the conceptual disadvantage that it renders at an intermediate stage numerous diagrams with different vertex degrees which have to be discarded at the end.

A more systematic and physical approach to construct all Feynman diagrams of a quantum field theory was proposed a long time ago [8, 9]. It is based on the observation that the complete knowledge of the vacuum energy implies the knowledge of the entire theory ("the vacuum is the world") [10, 11]. In this spirit, all vacuum diagrams are initially generated by a recursive graphical procedure. This procedure is derived from a functional differential equation involving functional derivatives with respect to free propagators and interactions. In a subsequent step, the n-point functions are found graphically by applying the functional derivatives to the vacuum energy. Recently, this approach was used to systematically generate all connected and one-particle irreducible Feynman diagrams

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of the euclidean multicomponent scalar ϕ^4-theory both in the disordered, symmetric phase [12] and in the ordered, spontaneously broken-symmetry phase [13, 14] (see also the related work in Ref. [15]). The approach was also applied to QED [16] and scalar QED [17] to construct the connected Feynman diagrams. In contrast to the conventional generating functional technique [18–23], no external currents coupled to single fields are used, such that there is no need for introducing Grassmann sources for fermion fields. An additional advantage is that the number of derivatives necessary to generate a certain correlation function is half as big as with external sources.

The purpose of the present paper is to modify the approach in Ref. [24] in such a way that it becomes applicable for QED. Rather than starting from vacuum diagrams as elaborated in Ref. [16], we generate the Feynman diagrams of n-point functions directly and find that they obey an infinite hierarchy of coupled Schwinger–Dyson equations [18–23]. We show that using functional derivatives with respect to the free propagators and the interaction allows us to close these Schwinger–Dyson equations functionally. In this way we obtain in Section II a closed set of equations determining the connected electron and photon two-point function as well as the connected three-point function. Analogously, we derive in Section III a closed set of equations for the electron and photon self-energy as well as the one-particle irreducible three-point function. In both cases, the closed set of Schwinger–Dyson equations can be converted into graphical recursion relations for the connected and one-particle irreducible Feynman diagrams. From these the corresponding vacuum diagrams follow by short-circuiting external legs.

II. CONNECTED FEYNMAN DIAGRAMS

Following the short-hand notation introduced in Ref. [16], the action of QED in euclidean spacetime reads

$$\mathcal{A}[\bar\psi,\psi,A] = \int_{12} S^{-1}_{12}\bar\psi_1\psi_2 + \frac{1}{2}\int_{12} D^{-1}_{12}A_1A_2 + \int_{123} V_{123}\bar\psi_1\psi_2A_3 - \int_1 J_1A_1, \tag{2.1}$$

where $\bar\psi$, ψ denote the electron fields and A stands for the photon field. For brevity, we omit all spinor or vector indices of the fields and indicate their spacetime arguments by simple number indices, i.e., $1 = x_1, 2 = x_2, \ldots,$ and $\int_1 = \int d^4x_1$. Throughout the paper we assume that the current J, the electron kernel S^{-1}, the photon kernel D^{-1}, and the interaction V are completely general non-singular functional matrices and their physical values for QED are inserted only at the end. By doing so, we regard the action (2.1) as the functional

$$\mathcal{A}[\bar\psi,\psi,A] = \mathcal{A}[\bar\psi,\psi,A; J, S^{-1}, D^{-1}, V]. \tag{2.2}$$

The same functional dependences are inherited by all field-theoretic quantities derived from it. In particular, we are interested in studying the functional dependence of the partition function, which is defined by a functional integral over a Boltzmann weight in natural units

$$Z[J, S^{-1}, D^{-1}, V] = \oint \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}A\, e^{-\mathcal{A}[\bar\psi,\psi,A;J,S^{-1},D^{-1},V]}, \tag{2.3}$$

and of the n-point functions

$$\begin{aligned}&\langle\psi_{n-2}\ldots\psi_4\psi_1\bar\psi_2\bar\psi_5\ldots\bar\psi_{n-1}A_3A_6\ldots A_n\rangle[J, S^{-1}, D^{-1}, V]\\ &\quad = \frac{1}{Z}\oint \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}A\,\psi_{n-2}\ldots\psi_4\psi_1\bar\psi_2\bar\psi_5\ldots\bar\psi_{n-1}A_3A_6\ldots A_n e^{-\mathcal{A}[\bar\psi,\psi,A;J,S^{-1},D^{-1},V]}.\end{aligned} \tag{2.4}$$

By expanding the functional integrals (2.3) and (2.4) in powers of the interaction V, the expansion coefficients of the partition function and the n-point functions consist of free-field expectation values. These are evaluated with the help of Wick's rule as a sum of Feynman integrals, which are pictured as diagrams constructed from lines and vertices. To illustrate the current we use the symbol

 $1 \equiv J_1 \,.$ (2.5)

As usual, a straight line with an arrow represents a free electron propagator

1 $2 \equiv S_{12},$ (2.6)

and a wiggly line indicates a free photon propagator

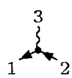
1 $2 \equiv D_{12}.$ (2.7)

Both propagators are inverse functional matrices of the corresponding kernels in the action (2.1):

$$\int_2 S_{12} S_{23}^{-1} = \delta_{13}, \tag{2.8}$$

$$\int_2 D_{12} D_{23}^{-1} = \delta_{13}. \tag{2.9}$$

A three-vertex represents the interaction potential:

 $\equiv -V_{123}.$ (2.10)

In this section we generate the subset of connected Feynman diagrams contributing to the partition function (2.3) and to the n-point functions (2.4) together with their weights. To this end we introduce in Section II.A functional derivatives with respect to the graphical elements J, S^{-1}, D^{-1}, V of the Feynman diagrams. With these we derive in Section II.B a closed set of Schwinger–Dyson equations determining the connected n-point functions. In Section II.C they are converted into graphical recursion relations for the corresponding connected Feynman diagrams. Finally the connected vacuum diagrams contributing to the vacuum energy are constructed in a graphical way in Section II.D.

A. Functional Derivatives

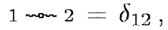
Each Feynman diagram of QED will be considered as a *functional* of the quantities characterizing the underlying field theory, the electron kernel S^{-1}, the photon kernel D^{-1}, the interaction V, and a current J. Following Refs. [12–14, 16, 17] we introduce in this subsection functional derivatives with respect to these, identify their associated graphical operation, and derive fundamental field-theoretic relations between them.

1. *Graphical Representation*

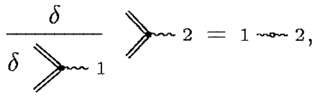
We start by studying the functional derivative with respect to the current J, which fulfills the identity

$$\frac{\delta J_2}{\delta J_1} = \delta_{12}. \tag{2.11}$$

We graphically represent the δ-function on the right-side by extending the elements of Feynman diagrams by an open dot with two labeled wiggly line ends


(2.12)

and picture the identity (2.11) graphically as


(2.13)

which leaves the spatial index at the line end to which the current was connected.

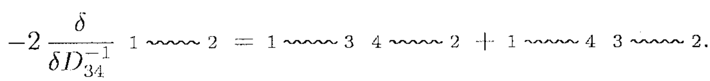
Since the photon field A is bosonic, the kernel D^{-1} is a symmetric functional matrix obeying $D^{-1}_{12} = D^{-1}_{21}$. This property is taken into account when performing functional derivatives with respect to the photon kernel D^{-1} with the basic rule

$$\frac{\delta D^{-1}_{12}}{\delta D^{-1}_{34}} = \frac{1}{2}\{\delta_{13}\delta_{42} + \delta_{14}\delta_{32}\}. \tag{2.14}$$

From the identity (2.9) and the functional chain rule of differentiation we find the derivative of the free propagator:

$$-2\,\frac{\delta D_{12}}{\delta D^{-1}_{34}} = D_{13}D_{42} + D_{14}D_{32}. \tag{2.15}$$

This has the graphical representation

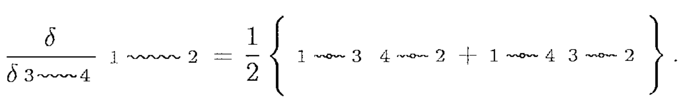

(2.16)

Thus, differentiating a photon propagator with respect to the kernel D^{-1} amounts to cutting the associated wiggly line into two pieces. The differentiation rule (2.14) ensures that the spatial indices of the kernel are symmetrically attached to the newly created line ends. When differentiating a general Feynman integral with respect to D^{-1}, the product rule of functional differentiation leads to a sum of diagrams in which each photon line is treated in this way.

We now study the graphical effect of functional derivatives with respect to the photon propagator D, where the basic differentiation rule reads

$$\frac{\delta D_{12}}{\delta D_{34}} = \frac{1}{2}\{\delta_{13}\delta_{42} + \delta_{14}\delta_{32}\}. \tag{2.17}$$

This is graphically written as


(2.18)

Thus differentiating a wiggly line with respect to the photon propagator removes the wiggly line, leaving in a symmetrized way the spatial indices of the wiggly line and the photon propagator.

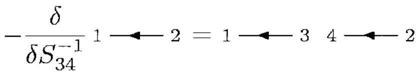
Setting up functional derivatives for electrons is different from the photon case, since the electron kernel S^{-1} is not symmetric. The functional derivative is simply

$$\frac{\delta S_{12}^{-1}}{\delta S_{34}^{-1}} = \delta_{13}\delta_{42}. \tag{2.19}$$

From a differentiation of the identity (2.8), we find

$$-\frac{\delta S_{12}}{\delta S_{34}^{-1}} = S_{13}S_{42}. \tag{2.20}$$

Its graphical representation



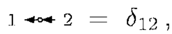
(2.21)

states that differentiating an electron propagator with respect to the kernel S^{-1} amouts to cutting the associated straight line with an arrow once.

As in (2.19) the functional derivative with respect to the electron propagator S reads

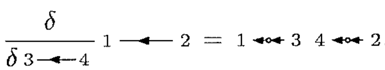
$$\frac{\delta S_{12}}{\delta S_{34}} = \delta_{13}\delta_{42}. \tag{2.22}$$

By analogy with (2.12), we represent a δ-function by an open dot with two labeled straight line ends with arrows



(2.23)

so that the differentiation rule (2.22) has the graphical form



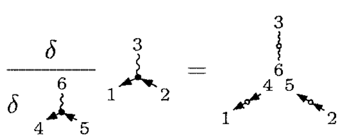
(2.24)

Thus differentiating an electron line with respect to the electron propagator removes the line, leaving the spatial indices of the electron propagator at the vertices to which the straight line with an arrow was connected.

The functional derivative with respect to the interaction V is defined by

$$\frac{\delta V_{123}}{\delta V_{456}} = \delta_{14}\delta_{25}\delta_{36}, \tag{2.25}$$

which has the graphical representation



(2.26)

Thus, differentiating a 3-vertex with respect to the interaction removes this vertex, leaving the spatial indices of the interaction at the line ends to which the vertex was connected.

2. *Field-Theoretic Identities*

With the help of these graphical operations, products of fields can be rewritten as functional derivatives of the action (2.1). Thus we obtain

$$A_1 = -\frac{\delta \mathcal{A}[\bar{\psi}, \psi, A]}{\delta J_1}, \tag{2.27}$$

$$\psi_1 \bar{\psi}_2 = -\frac{\delta \mathcal{A}[\bar{\psi}, \psi, A]}{\delta S^{-1}_{21}}, \tag{2.28}$$

$$A_1 A_2 = 2\frac{\delta \mathcal{A}[\bar{\psi}, \psi, A]}{\delta D^{-1}_{12}}, \tag{2.29}$$

$$\psi_1 \bar{\psi}_2 A_3 = -\frac{\delta \mathcal{A}[\bar{\psi}, \psi, A]}{\delta V_{213}}, \tag{2.30}$$

as follows from (2.11), (2.14), (2.19), and (2.25). Applying these derivatives to the integrands of the functional integrals (2.4) for the n-point functions, they can be determined from functional derivatives of the partition function (2.3) or its logarithm, the vacuum energy

$$W[J, S^{-1}, D^{-1}, V] = \ln Z[J, S^{-1}, D^{-1}, V]. \tag{2.31}$$

Thus we obtain the derivative rules

$$\langle A_1 \rangle = \frac{\delta W}{\delta J_1}, \tag{2.32}$$

$$\langle \psi_1 \bar{\psi}_2 \rangle = \frac{\delta W}{\delta S^{-1}_{21}}, \tag{2.33}$$

$$\langle A_1 A_2 \rangle = -2\frac{\delta W}{\delta D^{-1}_{12}}, \tag{2.34}$$

$$\langle \psi_1 \bar{\psi}_2 A_3 \rangle = \frac{\delta W}{\delta V_{213}}. \tag{2.35}$$

By doing so, we have to take into account compatibility relations between the different functional derivatives

$$\frac{\delta W}{\delta D^{-1}_{12}} = -\frac{1}{2}\left\{\frac{\delta^2 W}{\delta J_1 \delta J_2} + \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\right\}, \tag{2.36}$$

$$\frac{\delta W}{\delta V_{213}} = \frac{\delta^2 W}{\delta S^{-1}_{21} \delta J_3} + \frac{\delta W}{\delta S^{-1}_{21}}\frac{\delta W}{\delta J_3}, \tag{2.37}$$

which follow from the functional integral (2.3) for the partition function and (2.27)–(2.31). Thus there exist different ways of obtaining all diagrams of the n-point functions from the connected vacuum diagrams. From (2.34) and (2.36) we read off that, for instance, the diagrams of the photon two-point function follow either from cutting a photon line or from removing two currents of the connected vacuum diagrams in all possible ways. Consider as an example the vacuum energy for a vanishing interaction V, which follows directly from the functional integral according to (2.1), (2.3), and (2.31)

$$W^{(\text{free})} = W[J, S^{-1}, D^{-1}, 0] = \operatorname{Tr}\ln S^{-1} - \frac{1}{2}\operatorname{Tr}\ln D^{-1} + \frac{1}{2}\int_{12} D_{12}J_1J_2, \tag{2.38}$$

where the trace of the logarithm of a kernel $K^{-1} = S^{-1}, D^{-1}$ is defined by the series [25, p. 16]

$$\operatorname{Tr}\ln K^{-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_{1\ldots n}\left\{K_{12}^{-1} - \delta_{12}\right\}\cdots\left\{K_{n1}^{-1} - \delta_{n1}\right\}. \tag{2.39}$$

The free photon two-point function can be determined by applying functional derivatives to (2.38) either with respect to the photon kernel D^{-1} or with respect to the current J. In both cases we obtain

$$\langle A_1A_2\rangle^{(\text{free})} = D_{12} + \int_{34} D_{13}D_{24}J_3J_4. \tag{2.40}$$

Relations similar to (2.32)–(2.35) follow for the connected n-point functions which are defined as

$$\boldsymbol{A}_1^{\text{c}} = \langle A_1\rangle, \tag{2.41}$$

$$\boldsymbol{S}_{12}^{\text{c}} = \langle\psi_1\bar{\psi}_2\rangle, \tag{2.42}$$

$$\boldsymbol{D}_{12}^{\text{c}} = \langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle, \tag{2.43}$$

$$\boldsymbol{G}_{123}^{\text{c}} = \langle\psi_1\bar{\psi}_2A_3\rangle - \langle\psi_1\bar{\psi}_2\rangle\langle A_3\rangle, \tag{2.44}$$

resulting in the following derivative rules for W:

$$\boldsymbol{A}_1^{\text{c}} = \frac{\delta W}{\delta J_1}, \tag{2.45}$$

$$\boldsymbol{S}_{12}^{\text{c}} = \frac{\delta W}{\delta S_{21}^{-1}}, \tag{2.46}$$

$$\boldsymbol{D}_{12}^{\text{c}} = -2\,\frac{\delta W}{\delta D_{12}^{-1}} - \boldsymbol{A}_1^{\text{c}}\boldsymbol{A}_2^{\text{c}}, \tag{2.47}$$

$$\boldsymbol{G}_{123}^{\text{c}} = \frac{\delta W}{\delta V_{213}} - \boldsymbol{S}_{12}^{\text{c}}\boldsymbol{A}_3^{\text{c}}. \tag{2.48}$$

From (2.47) we read off that, for instance, cutting a photon line of the connected vacuum diagrams in all possible ways also leads to connected and disconnected pieces. The latter are removed by the term $\boldsymbol{A}_1^{\text{c}}\boldsymbol{A}_2^{\text{c}}$, leading to the diagrams contributing to the connected photon two-point function $\boldsymbol{D}_{12}^{\text{c}}$. For later purposes we note that the connected three-point function (2.48) may be rewritten as

$$\boldsymbol{G}_{123}^{\text{c}} = \frac{\delta^2 W}{\delta S_{21}^{-1}\delta J_3}. \tag{2.49}$$

This follows from the compatibility relation (2.37) as well as from (2.45) and (2.46).

B. Closed Set of Schwinger–Dyson Equations for Connected n-Point Functions

We now apply the above functional derivatives to certain trivial functional identities which immediately follow from the definition of the functional integral. By doing so, we derive a closed set of functional equations determining the connected electron and photon two-point function as well as the connected three-point function.

1. *Connected Electron Two-Point Function*

Consider the functional identity

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \frac{\delta}{\delta\bar{\psi}_1}\{\bar{\psi}_2\, e^{-\mathcal{A}[\bar{\psi},\psi,A;J,S^{-1},D^{-1},V]}\} = 0, \tag{2.50}$$

which follows by functional integration from the vanishing of the exponential at infinite fields. Performing the functional derivative in the integrand and taking into account the action (2.1) leads to

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \left\{\delta_{12} + \int_3 S_{13}^{-1}\bar{\psi}_2\psi_3 + \int_{34} V_{134}\bar{\psi}_2\psi_3 A_4\right\} e^{-\mathcal{A}[\bar{\psi},\psi,A;J,S^{-1},D^{-1},V]} = 0. \tag{2.51}$$

Replacing the fields A_4 and $\bar{\psi}_2\psi_3$ by functional derivatives with respect to the current J_4 and the electron kernel S_{23}^{-1} using (2.27) and (2.28), respectively, the equation can be expressed in terms of the vacuum energy W by taking into account (2.3) and (2.31):

$$\delta_{12} - \int_3 S_{13}^{-1}\frac{\delta W}{\delta S_{23}^{-1}} - \int_{34} V_{134}\left\{\frac{\delta^2 W}{\delta S_{23}^{-1}\delta J_4} + \frac{\delta W}{\delta S_{23}^{-1}}\frac{\delta W}{\delta J_4}\right\} = 0. \tag{2.52}$$

The functional derivative with respect to the current J in the last term can be eliminated with the help of the second functional identity

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \frac{\delta}{\delta A_1} e^{-\mathcal{A}[\bar{\psi},\psi,A;J,S^{-1},D^{-1},V]} = 0. \tag{2.53}$$

After differentiating the action (2.1) in the exponential, we find

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \left\{-J_1 + \int_2 D_{12}^{-1}A_2 + \int_{23} V_{231}\bar{\psi}_2\psi_3\right\} e^{-\mathcal{A}[\bar{\psi},\psi,A;J,S^{-1},D^{-1},V]} = 0, \tag{2.54}$$

which leads to

$$\frac{\delta W}{\delta J_4} = \int_5 D_{45}J_5 + \int_{567} V_{567}D_{47}\frac{\delta W}{\delta S_{56}^{-1}}. \tag{2.55}$$

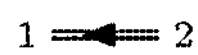
Inserting this into (2.52), we obtain

$$\delta_{12} - \int_3 S_{13}^{-1}\frac{\delta W}{\delta S_{23}^{-1}} - \int_{34} V_{134}\frac{\delta^2 W}{\delta S_{23}^{-1}\delta J_4} - \int_{345} V_{134}D_{45}J_5\frac{\delta W}{\delta S_{23}^{-1}} - \int_{34567} V_{134}V_{567}D_{47}\frac{\delta W}{\delta S_{23}^{-1}}\frac{\delta W}{\delta S_{56}^{-1}} = 0. \tag{2.56}$$

Taking into account the definitions of the connected electron two-point function (2.46) and the connected three-point function (2.49), this equation reduces to the Schwinger–Dyson equation for $\boldsymbol{S}^{\mathrm{c}}$:

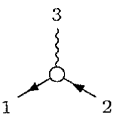
$$\boldsymbol{S}^{\mathrm{c}}_{12} = S_{12} - \int_{345} V_{345} S_{13} \boldsymbol{G}^{\mathrm{c}}_{425} - \int_{345678} V_{345} V_{678} S_{13} D_{58} \boldsymbol{S}^{\mathrm{c}}_{42} \boldsymbol{S}^{\mathrm{c}}_{76} - \int_{3456} V_{345} D_{56} J_6 \boldsymbol{S}^{\mathrm{c}}_{42} S_{13}. \quad (2.57)$$

In order to represent this graphically, we extend the elements of Feynman diagrams by a symbol for the fully interacting connected electron two-point function



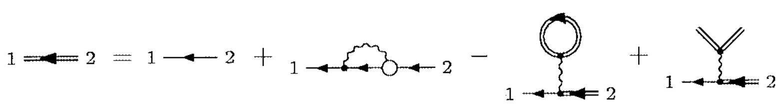
$$\equiv \boldsymbol{S}^{\mathrm{c}}_{12}\,, \quad (2.58)$$

and a three-vertex with an open dot representing the fully interacting connected three-point function



$$\equiv \boldsymbol{G}^{\mathrm{c}}_{123}\,. \quad (2.59)$$

With this, the Schwinger–Dyson equation (2.57) reads graphically



(2.60)

2. *Connected Photon Two-Point Function*

Now we determine in a similar way the connected photon two-point function. To this end we consider the third functional identity

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \, \frac{\delta}{\delta A_1}\left\{A_2 e^{-\mathcal{A}[\bar{\psi},\psi,A;J,S^{-1},D^{-1},V]}\right\} = 0, \quad (2.61)$$

which leads to

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \left\{\delta_{12} + A_2 J_1 - \int_3 D^{-1}_{13} A_2 A_3 - \int_{34} V_{341} A_2 \bar{\psi}_3 \psi_4\right\} e^{-\mathcal{A}[\bar{\psi},\psi,A;J,S^{-1},D^{-1},V]} = 0. \quad (2.62)$$

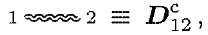
Substituting products of fields according to Eqs. (2.27), (2.29), and (2.30), and using (2.37), we obtain

$$\delta_{12} + J_1 \frac{\delta W}{\delta J_2} + 2\int_3 D^{-1}_{13} \frac{\delta W}{\delta D^{-1}_{23}} + \int_{34} V_{341} \left\{\frac{\delta^2 W}{\delta S^{-1}_{34} \delta J_2} + \frac{\delta W}{\delta S^{-1}_{34}} \frac{\delta W}{\delta J_2}\right\} = 0. \quad (2.63)$$

Taking into account the definitions of the connected photon two-point function (2.47), the connected electron two-point function (2.46), and the connected three-point function (2.49), the functional derivative with respect to the current J is eliminated by using (2.55). In this way we result in the Schwinger–Dyson equation determining $\boldsymbol{D}^{\mathrm{c}}$:

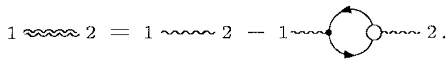
$$\boldsymbol{D}^{\mathrm{c}}_{12} = D_{12} + \int_{345} V_{345} \boldsymbol{G}^{\mathrm{c}}_{432} D_{15}. \quad (2.64)$$

Extending the elements of Feynman diagrams by a symbol for the fully interacting connected photon two-point function



(2.65)

this Schwinger–Dyson equation reads graphically



(2.66)

3. *Connected Three-Point Function*

The iteration of the integral equations (2.60) and (2.66) for the connected electron and photon two-point function $\boldsymbol{S}^{c}$ and $\boldsymbol{D}^{c}$ requires the knowledge of the connected three-point function $\boldsymbol{G}^{c}$. Therefore we evaluate (2.49) further by inserting (2.55):

$$\boldsymbol{G}^{c}_{123} = \int_{456} V_{456} D_{36} \frac{\delta^{2} W}{\delta S^{-1}_{21} \delta S^{-1}_{45}}. \tag{2.67}$$

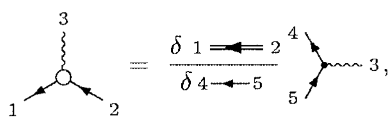
Taking into account the definition of the connected electron two-point function (2.46) and the functional chain rule

$$\frac{\delta}{\delta S^{-1}_{12}} = -\int_{34} S_{31} S_{24} \frac{\delta}{\delta S_{34}}, \tag{2.68}$$

this equation leads to a functional integrodifferential equation for the connected three-point function

$$\boldsymbol{G}^{c}_{123} = -\int_{45678} V_{456} D_{36} S_{74} S_{58} \frac{\delta \boldsymbol{S}^{c}_{12}}{\delta S_{78}}. \tag{2.69}$$

Its graphical representation reads

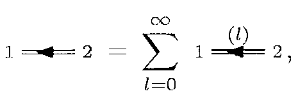


(2.70)

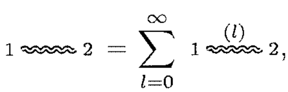
so that the diagrams of the connected three-point function follow from those of the connected electron two-point function by inserting a three-vertex in an electron line in all possible ways. Thus the closed set of Schwinger–Dyson equations is given by (2.60), (2.66), and (2.70).

C. Graphical Recursion Relations

Now we demonstrate how the diagrams of the connected electron and photon two-point function as well as of the connected three-point function are recursively generated in a graphical way. To simplify the discussion we restrict ourselves to the case of a vanishing external current, so that we can neglect the last term in (2.60). Performing a loop expansion of the connected electron and photon two-point function

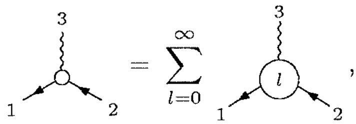


(2.71)



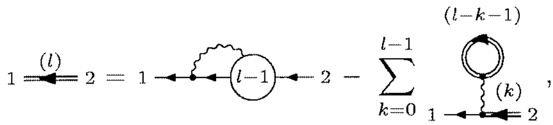
(2.72)

as well as for the connected three-point function

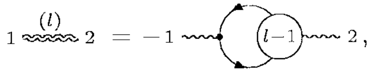


(2.73)

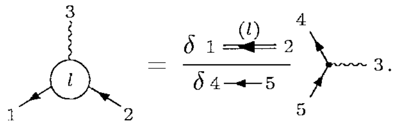
we obtain from (2.60), (2.66), and (2.70) the following closed set of graphical recursion relations:



(2.74)

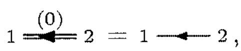


(2.75)



(2.76)

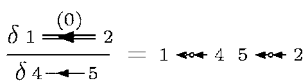
This is solved starting from



(2.77)

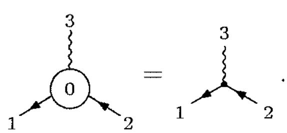
(2.78)

At first we evaluate the amputation of one electron line from (2.77),



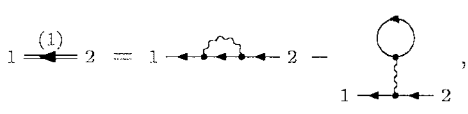
(2.79)

and insert this into (2.76) to obtain the connected three-point function for $l = 0$:

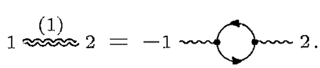


(2.80)

With this we get from (2.74) and (2.75) the one-loop contribution to the connected electron and photon two-point function:

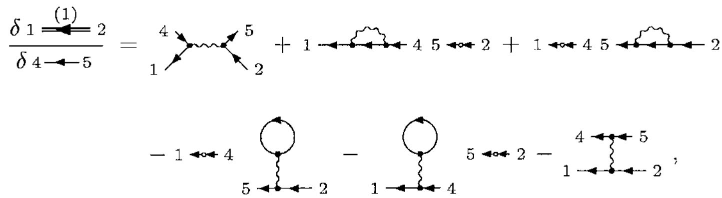


(2.81)



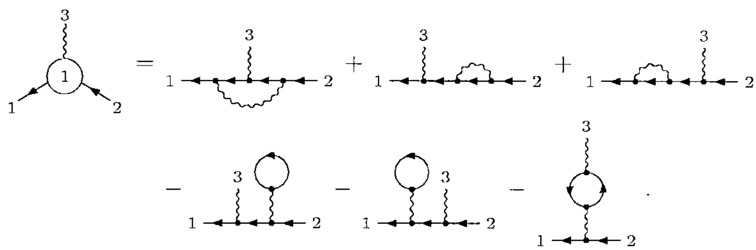
(2.82)

Amputating one electron line from (2.81),



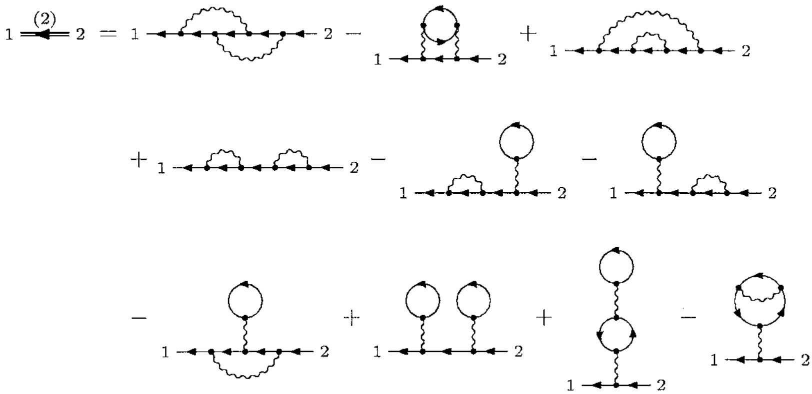
(2.83)

we find the one-loop contribution to the connected three-point function:



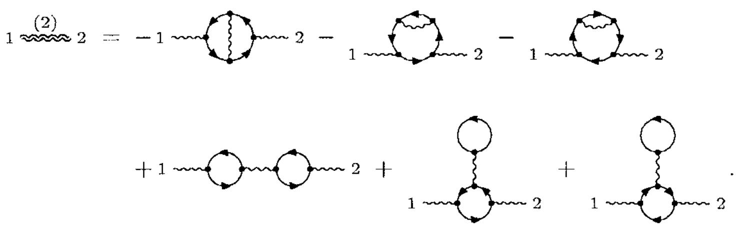
(2.84)

Thus we obtain from (2.74) and (2.75) the two-loop connected two-point function of the electron



(2.85)

and of the photon



(2.86)

From the Feynman diagrams of the connected electron and photon two-point function as well as the connected three-point function generated so far, we read off a simple rule for their weights. They are given by $(-1)^l$ with l being the number of electron loops [16]. The same Feynman diagrams have been obtained in Ref. [16] by amputating lines or vertices in the connected vacuum diagrams.

D. Connected Vacuum Diagrams

The connected vacuum diagrams of QED can be generated together with their weights in two different ways. First, they can be constructed from the above diagrams of the connected electron and photon two-point function as well as the connected three-point function. Second, we derive from the above functional identities a nonlinear functional differential equation for the vacuum energy and convert it into a graphical recursion relation which directly generates the connected vacuum diagrams as in Ref. [16].

1. *Relation to the Diagrams of the Connected n-Point Functions*

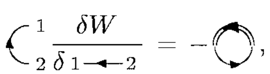
After having iteratively solved the closed set of graphical recursion relations (2.74)–(2.76) for the diagrams of the connected electron and photon two-point function as well as of the connected three-point function, the corresponding connected vacuum diagrams can be constructed loopwise as follows. Going back to the defining equations (2.46)–(2.48) for $\boldsymbol{S}^{\mathrm{c}}$, $\boldsymbol{D}^{\mathrm{c}}$, $\boldsymbol{G}^{\mathrm{c}}$, we obtain with (2.45), (2.55), and the functional chain rule (2.68) three functional differential equations for the vacuum energy:

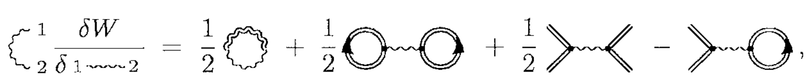
$$\int_{12} S_{12} \frac{\delta W}{\delta S_{12}} = -\int_{12} S_{21}^{-1} \boldsymbol{S}_{12}^{\mathrm{c}}, \tag{2.87}$$

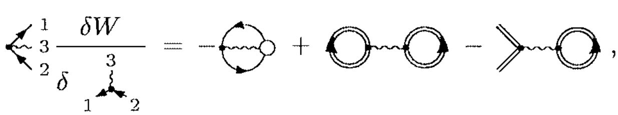
$$\int_{12} D_{12} \frac{\delta W}{\delta D_{12}} = \frac{1}{2} \int_{12} D_{12}^{-1} \boldsymbol{D}_{12}^{\mathrm{c}} + \frac{1}{2} \int_{123456} V_{123} V_{456} \boldsymbol{S}_{21}^{\mathrm{c}} \boldsymbol{S}_{54}^{\mathrm{c}} D_{36}$$
$$+ \frac{1}{2} \int_{12} D_{12} J_1 J_2 + \int_{1234} V_{123} \boldsymbol{S}_{21}^{\mathrm{c}} D_{34} J_4, \tag{2.88}$$

$$\int_{123} V_{123} \frac{\delta W}{\delta V_{123}} = \int_{123} V_{123} \boldsymbol{G}_{213}^{\mathrm{c}} + \int_{123456} V_{123} V_{456} \boldsymbol{S}_{21}^{\mathrm{c}} \boldsymbol{S}_{54}^{\mathrm{c}} D_{36} + \int_{1234} V_{123} \boldsymbol{S}_{21}^{\mathrm{c}} D_{34} J_4. \tag{2.89}$$

Their graphical representations are

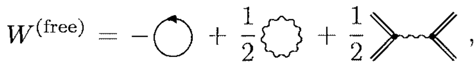

(2.90)


(2.91)


(2.92)

where the first term on the right-hand side of (2.90) and (2.91) pictures the closing of the external legs of the connected electron and photon two-point function, respectively. All three equations have in common that the terms on the left-hand side count the number of a graphical element of each connected vacuum diagram. Indeed, when performing the operation $\int E\delta/\delta E$ with $E = S, D, V$, the functional derivative $\delta/\delta E$ removes successively an electron line, a photon line, or a three-vertex in all possible ways, which is subsequently reinserted by the operation $\int E$.

If the interaction V vanishes, Eqs. (2.90)–(2.92) are solved by the free contribution to the vacuum energy (2.38) with the graphical representation



(2.93)

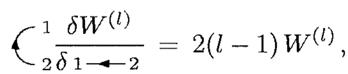
due to (2.77) and (2.78). For a non-vanishing interaction V, Eqs. (2.90)–(2.92) produce corrections to (2.93) which we shall denote with $W^{(\mathrm{int})}$. Thus the vacuum energy W decomposes according to

$$W = W^{(\mathrm{free})} + W^{(\mathrm{int})}. \tag{2.94}$$

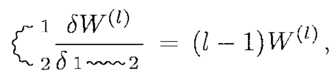
In the following we recursively determine $W^{(\mathrm{int})}$ in a graphical way for a vanishing external current, so that we can neglect the last two terms in (2.91) and the last term in (2.92). Performing a loopwise expansion of the interaction part of the vacuum energy

$$W^{(\mathrm{int})} = \sum_{l=2}^{\infty} W^{(l)}, \tag{2.95}$$

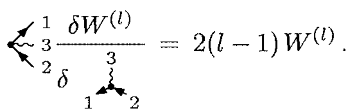
we use the following eigenvalue problems for $l \geq 2$:



(2.96)

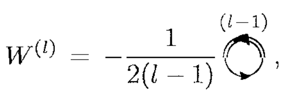


(2.97)

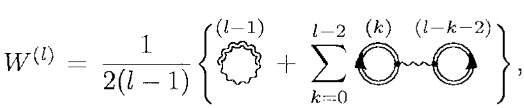


(2.98)

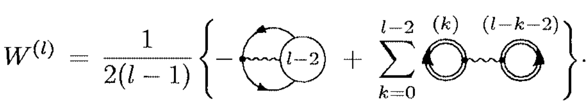
With these we explicitly solve (2.90)–(2.92) for the expansion coefficients $W^{(l)}$ and obtain for $l \geq 2$



(2.99)

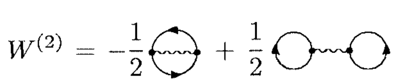


(2.100)



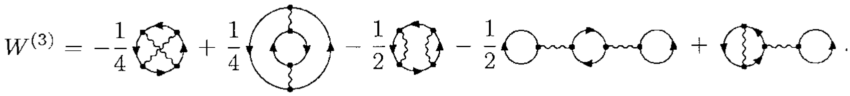
(2.101)

Inserting (2.77)–(2.86) for the lower loop contributions of the connected electron and photon two-point function as well as the connected three-point function in one of Eqs. (2.99)–(2.101), we find the vacuum energy for two loops



(2.102)

and for three loops



(2.103)

2. *Graphical Recursion Relation*

Each of the three functional differential equations for the vacuum energy (2.87)–(2.89) can be used to derive a graphical recursion relation which directly leads to the connected vacuum diagrams. Here we restrict ourselves to the functional differential equation (2.87) which is based on counting the number of electron lines of the connected vacuum diagrams. Inserting Eqs. (2.46), (2.57), and (2.69) for the connected electron two-point function and the connected three-point function, we obtain from (2.87) via the functional chain rule (2.68)

$$\delta_{11}\int_1 - \int_{12} S^{-1}_{12}\frac{\delta W}{\delta S^{-1}_{12}} = \int_{123456} V_{123}V_{456}D_{36}\left\{\frac{\delta^2 W}{\delta S^{-1}_{12}\,\delta S^{-1}_{45}} + \frac{\delta W}{\delta S^{-1}_{12}}\,\frac{\delta W}{\delta S^{-1}_{45}}\right\} + \int_{1234} V_{123}D_{34}J_4\frac{\delta W}{\delta S^{-1}_{12}}. \tag{2.104}$$

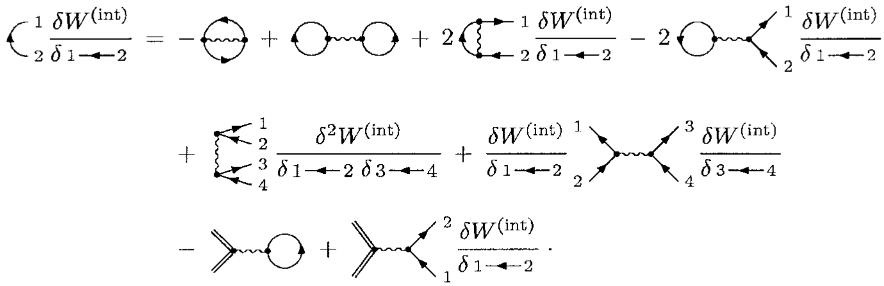
If the interaction V vanishes, this equation is solved by the free vacuum energy (2.38) which has the functional derivatives

$$\frac{\delta W^{(\text{free})}}{\delta S^{-1}_{12}} = S_{21}, \qquad \frac{\delta^2 W^{(\text{free})}}{\delta S^{-1}_{12}\,\delta S^{-1}_{45}} = -S_{24}S_{51}. \tag{2.105}$$

For a non-vanishing interaction V, the right-hand side of (2.104) corrects (2.38) by the interaction part of the vacuum energy $W^{(\text{int})}$. Inserting the decomposition (2.94) into (2.104), and using (2.105), we obtain together with the functional chain rule (2.68) the following functional differential equation for the interaction part of the vacuum energy:

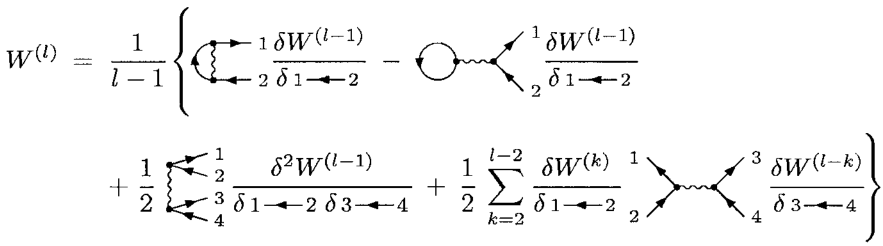
$$\begin{aligned}
\int_{12} S_{12}\frac{\delta W^{(\text{int})}}{\delta S_{12}} = {} & \int_{123456} V_{123}V_{456}S_{21}S_{54}D_{36} - \int_{123456} V_{123}V_{456}S_{24}S_{51}D_{36} \\
& - 2\int_{12345678} V_{123}V_{456}D_{36}S_{21}S_{74}S_{58}\frac{\delta W^{(\text{int})}}{\delta S_{78}} \\
& + 2\int_{12345678} V_{123}V_{456}D_{36}S_{51}S_{28}S_{74}\frac{\delta W^{(\text{int})}}{\delta S_{78}} \\
& + \int_{123456789\bar{1}} V_{123}V_{456}D_{36}S_{71}S_{28}S_{94}S_{5\bar{1}}\frac{\delta W^{(\text{int})}}{\delta S_{78}}\,\frac{\delta W^{(\text{int})}}{\delta S_{9\bar{1}}} \\
& + \int_{123456789\bar{1}} V_{123}V_{456}D_{36}S_{71}S_{28}S_{94}S_{5\bar{1}}\frac{\delta^2 W^{(\text{int})}}{\delta S_{78}S_{9\bar{1}}} + \int_{1234} V_{123}D_{34}S_{21}J_4 \\
& - \int_{123456} V_{123}S_{51}S_{26}D_{34}J_4\frac{\delta W^{(\text{int})}}{\delta S_{56}}.
\end{aligned} \tag{2.106}$$

Its graphical representation is



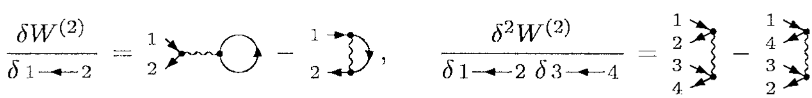
(2.107)

As before, we illustrate the graphical recursive solution only for a vanishing external current so that we can drop the last two terms in (2.107). Thus inserting the loop expansion (2.95) and using the eigenvalue problem (2.96), we obtain a graphical recursion relation for the expansion coefficients $W^{(l)}$ of the vacuum energy for $l \geq 3$ [16]:



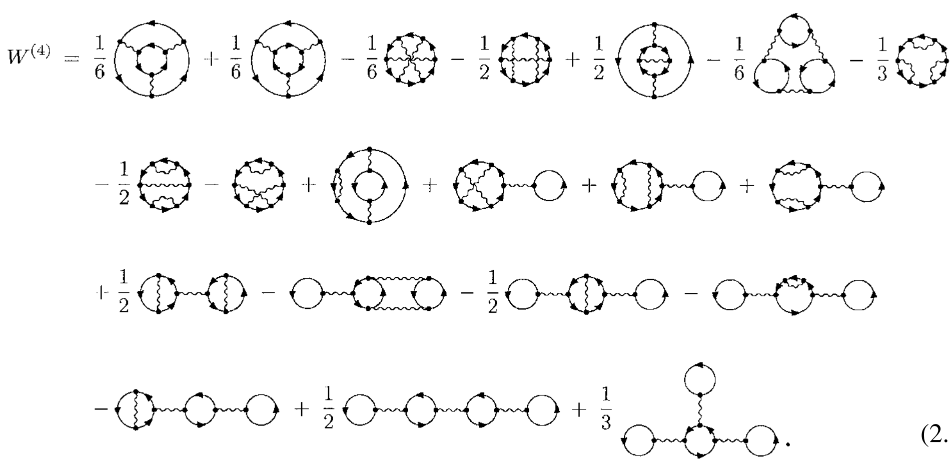
(2.108)

They are solved starting from $W^{(2)}$ in (2.102). We start with the amputation of one or two electron lines in (2.102):



(2.109)

Inserting (2.109) into (2.108), we reobtain the three-loop contribution of the vacuum energy from (2.103). The corresponding calculation of the four-loop correction $W^{(4)}$ leads altogether to 20 connected vacuum diagrams:



(2.110)

From the vacuum diagrams (2.102), (2.103), and (2.110), we observe a simple mnemonic rule for the weights of the connected vacuum diagrams in QED [16]. At least up to four loops, each weight is equal to the reciprocal number of electron lines, which are generated by cutting the same electron two-point diagrams. The sign is given by $(-1)^l$, where l denotes the number of electron loops. Let us also note that the total weight, which is the sum of all weights of the connected vacuum diagrams in the loop order under considertation, vanishes in QED. These simple weights are a consequence of the Fermi statistics and the three-vertex of the interaction in (2.1). The weights of the vacuum diagrams in other theories, like ϕ^4-theory [12–14, 26], follow more complicated rules.

III. ONE-PARTICLE IRREDUCIBLE FEYNMAN DIAGRAMS

So far, we have generated all connected Feynman diagrams of QED. We now eliminate the one-particle reducible Feynman diagrams. This is done as usual with the help of a functional Legendre transform with respect to the current which we introduce in Section III.A. With this we derive in Section III.B a closed set of Schwinger–Dyson equations for the one-particle irreducible n-point functions. In Section III.C they are converted into graphical recursion relations for the corresponding one-particle irreducible Feynman diagrams needed for renormalizing QED. Finally, the one-particle irreducible vacuum diagrams are constructed graphically in Section III.D.

A. Functional Legendre Transform with Respect to the Current

We set up the functional Legendre transform with respect to the current which converts the vacuum energy W to the effective energy of the first kind Γ_1. In particular, we investigate the respective functional derivatives of W and Γ_1 and the field-theoretic relations between them.

1. *Effective Energy of the First Kind*

Starting from the vacuum energy $W[J, S^{-1}, D^{-1}, V]$ we introduce the new field

$$\boldsymbol{A}_1^{\mathrm{c}}[J, S^{-1}, D^{-1}, V] = \left.\frac{\delta W[J, S^{-1}, D^{-1}, V]}{\delta J_1}\right|_{S^{-1}, D^{-1}, V}, \tag{3.1}$$

which implicitly defines J as a functional of $\boldsymbol{A}^{\mathrm{c}}$:

$$J_1 = J_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]. \tag{3.2}$$

From Eq. (2.45) we read off that $\boldsymbol{A}^{\mathrm{c}}$ coincides with the field expectation value of the photon field in the presence of the current J. The functional Legendre transform of the vacuum energy $W[J, S^{-1}, D^{-1}, V]$ with respect to the current J results in the effective energy of the first kind

$$\Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V] = \int_1 J_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V] \left.\frac{\delta W[J[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V], S^{-1}, D^{-1}, V]}{\delta J_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}\right|_{S^{-1}, D^{-1}, V}$$
$$- W[J[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V], S^{-1}, D^{-1}, V], \tag{3.3}$$

which simplifies due to (3.1):

$$\Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V] = \int_1 J_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]\boldsymbol{A}_1^{\mathrm{c}} - W[J[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V], S^{-1}, D^{-1}, V]. \tag{3.4}$$

Taking into account the functional chain rule, it leads to the equation of state

$$\left.\frac{\delta\Gamma_1[\boldsymbol{A}^{\rm c}, S^{-1}, D^{-1}, V]}{\delta \boldsymbol{A}_1^{\rm c}}\right|_{S^{-1},D^{-1},V} = J_1[\boldsymbol{A}^{\rm c}, S^{-1}, D^{-1}, V]. \tag{3.5}$$

Performing a loop expansion, the contributions to the effective energy of the first kind (3.4) may be displayed as one-particle irreducible vacuum diagrams which are constructed according to the Feynman rules (2.6), (2.7), and (2.10). In addition, a dot with a wiggly line represents the field expectation value of the photon

$$1 \equiv \boldsymbol{A}_1^{\rm c}. \tag{3.6}$$

If the interaction V vanishes, the vacuum energy (2.38) leads with (3.1) to the field expectation value

$$\boldsymbol{A}_1^{\rm c}[J, S^{-1}, D^{-1}, 0] = \int_2 D_{12} J_2, \tag{3.7}$$

which is inverted to give

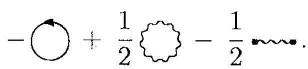
$$J_1[\boldsymbol{A}^{\rm c}, S^{-1}, D^{-1}, 0] = \int_2 D_{12}^{-1} \boldsymbol{A}_2^{\rm c}, \tag{3.8}$$

leading to the free effective energy of the first kind:

$$\Gamma_1^{(\rm free)} = \Gamma_1[\boldsymbol{A}^{\rm c}, S^{-1}, D^{-1}, 0] = -\mathrm{Tr}\ln S^{-1} + \frac{1}{2}\mathrm{Tr}\ln D^{-1} + \frac{1}{2}\int_{12} D_{12}^{-1} \boldsymbol{A}_1^{\rm c} \boldsymbol{A}_2^{\rm c}. \tag{3.9}$$

Its graphical representation is

$$-\Gamma_1^{(\rm free)} = \tag{3.10}$$



In order to investigate in detail the field-theoretic consequences of the functional Legendre transform of the first kind, it is advantageous to start with the effective energy of the first kind $\Gamma_1[\boldsymbol{A}^{\rm c}, S^{-1}, D^{-1}, V]$ and to introduce the current J via the equation of state (3.5). This implicitly defines the field expectation value of the photon as a functional of the current, i.e.,

$$\boldsymbol{A}_1^{\rm c} = \boldsymbol{A}_1^{\rm c}[J, S^{-1}, D^{-1}, V]. \tag{3.11}$$

Thus the vacuum energy is recovered by the inverse functional Legendre transform

$$W[J, S^{-1}, D^{-1}, V] = \int_1 J_1 \boldsymbol{A}_1^{\rm c}[J, S^{-1}, D^{-1}, V] - \Gamma_1[\boldsymbol{A}^{\rm c}[J, S^{-1}, D^{-1}, V], S^{-1}, D^{-1}, V]. \tag{3.12}$$

With this we derive useful relations between the functional derivatives of the vacuum energy W and the effective energy of the first kind Γ_1, respectively.

2. *Functional Derivatives*

Taking into account the functional chain rule, the first functional derivatives of the vacuum energy W read (3.1) and

$$\left.\frac{\delta W[J,S^{-1},D^{-1},V]}{\delta S^{-1}_{12}}\right|_{J,D^{-1},V} = -\left.\frac{\delta\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta S^{-1}_{12}}\right|_{\boldsymbol{A}^{\rm c},D^{-1},V}, \tag{3.13}$$

$$\left.\frac{\delta W[J,S^{-1},D^{-1},V]}{\delta D^{-1}_{12}}\right|_{J,S^{-1},V} = -\left.\frac{\delta\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta D^{-1}_{12}}\right|_{\boldsymbol{A}^{\rm c},S^{-1},V}, \tag{3.14}$$

$$\left.\frac{\delta W[J,S^{-1},D^{-1},V]}{\delta V_{123}}\right|_{J,S^{-1},D^{-1}} = -\left.\frac{\delta\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta V_{123}}\right|_{\boldsymbol{A}^{\rm c},S^{-1},D^{-1}}. \tag{3.15}$$

To evaluate second functional derivatives of the vacuum energy W is more involved. First, we observe

$$\begin{aligned}\left.\frac{\delta^2 W[J,S^{-1},D^{-1},V]}{\delta J_2\delta J_1}\right|_{S^{-1},D^{-1},V} &= \left.\frac{\delta \boldsymbol{A}^{\rm c}_1[J,S^{-1},D^{-1},V]}{\delta J_2}\right|_{S^{-1},D^{-1},V}\\ &= \left(\left.\frac{\delta J_2[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta \boldsymbol{A}^{\rm c}_1[J,S^{-1},D^{-1},V]}\right|_{S^{-1},D^{-1},V}\right)^{-1}\\ &= \left(\left.\frac{\delta^2\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta \boldsymbol{A}^{\rm c}_1[J,S^{-1},D^{-1},V]\delta \boldsymbol{A}^{\rm c}_2[J,S^{-1},D^{-1},V]}\right|_{S^{-1},D^{-1},V}\right)^{-1},\end{aligned} \tag{3.16}$$

where we have used (3.1), (3.5), and the fact that the derivative of a functional equals the inverse of the derivative of the inverse functional. To make precise the meaning of relation (3.16), we rederive it from another point of view. Considering the functional identity

$$\left.\frac{\delta J_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta J_2}\right|_{S^{-1},D^{-1},V} = \delta_{12}, \tag{3.17}$$

we apply the functional chain rule together with (3.1) and (3.5). Thus we result in

$$\int_3 \left.\frac{\delta^2\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta \boldsymbol{A}^{\rm c}_1[J,S^{-1},D^{-1},V]\delta \boldsymbol{A}^{\rm c}_3[J,S^{-1},D^{-1},V]}\right|_{S^{-1},D^{-1},V} \left.\frac{\delta^2 W[J,S^{-1},D^{-1},V]}{\delta J_3\delta J_2}\right|_{S^{-1},D^{-1},V} = \delta_{12}, \tag{3.18}$$

which coincides with (3.16). Furthermore we obtain from (3.13) by applying again the functional chain rule and relation (3.16)

$$\begin{aligned}&\frac{\delta}{\delta J_3}\left(\left.\frac{\delta W[J,S^{-1},D^{-1},V]}{\delta S^{-1}_{12}}\right|_{J,D^{-1},V}\right)_{S^{-1},D^{-1},V}\\ &= -\int_4\left(\left.\frac{\delta^2\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta \boldsymbol{A}^{\rm c}_3[J,S^{-1},D^{-1},V]\delta \boldsymbol{A}^{\rm c}_4[J,S^{-1},D^{-1},V]}\right|_{S^{-1},D^{-1},V}\right)^{-1}\\ &\quad\times\frac{\delta}{\delta \boldsymbol{A}^{\rm c}_4[J,S^{-1},D^{-1},V]}\left(\left.\frac{\delta\Gamma_1[\boldsymbol{A}^{\rm c}[J,S^{-1},D^{-1},V],S^{-1},D^{-1},V]}{\delta S^{-1}_{12}}\right|_{\boldsymbol{A}^{\rm c},D^{-1},V}\right)_{S^{-1},D^{-1},V}.\end{aligned} \tag{3.19}$$

3. *Field-Theoretic Identities*

Performing the functional Legendre transform with respect to the current, the compatibility relation (2.36) between functional derivatives with respect to the current J and the photon kernel D^{-1} yields

$$\left(\left. \frac{\delta^2 \Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}{\delta \boldsymbol{A}^{\mathrm{c}}_2 \delta \boldsymbol{A}^{\mathrm{c}}_1} \right|_{S^{-1}, D^{-1}, V} \right)^{-1} = 2 \left. \frac{\delta \Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}{\delta D^{-1}_{12}} \right|_{\boldsymbol{A}^{\mathrm{c}}, D^{-1}, V} - \boldsymbol{A}^{\mathrm{c}}_1 \boldsymbol{A}^{\mathrm{c}}_2 \tag{3.20}$$

due to (3.1), (3.14), and (3.16). In a similar way, the compatibility relation (2.37) between functional derivatives with respect to the current J, the electron kernel S^{-1}, and the interaction V is converted using (3.1), (3.13), (3.15), and (3.19) to

$$\int_4 \frac{\delta}{\delta \boldsymbol{A}^{\mathrm{c}}_4} \left(\left. \frac{\delta \Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}{\delta S^{-1}_{12}} \right|_{\boldsymbol{A}^{\mathrm{c}}, D^{-1}, V} \right)_{S^{-1}, D^{-1}, V} \left(\left. \frac{\delta^2 \Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}{\delta \boldsymbol{A}^{\mathrm{c}}_3 \delta \boldsymbol{A}^{\mathrm{c}}_4} \right|_{S^{-1}, D^{-1}, V} \right)^{-1}$$
$$= \left. \frac{\delta \Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}{\delta V_{123}} \right|_{\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}} - \left. \frac{\delta \Gamma_1[\boldsymbol{A}^{\mathrm{c}}, S^{-1}, D^{-1}, V]}{\delta S^{-1}_{12}} \right|_{\boldsymbol{A}^{\mathrm{c}}, D^{-1}, V} \boldsymbol{A}^{\mathrm{c}}_3. \tag{3.21}$$

The functional Legendre transform has also consequences for the connected n-point functions (2.46)–(2.48). Taking into account (3.1) and (3.13)–(3.15), we obtain

$$\boldsymbol{S}^{\mathrm{c}}_{12} = -\frac{\delta \Gamma_1}{\delta S^{-1}_{21}}, \tag{3.22}$$

$$\boldsymbol{D}^{\mathrm{c}}_{12} = 2 \frac{\delta \Gamma_1}{\delta D^{-1}_{12}} - \boldsymbol{A}^{\mathrm{c}}_1 \boldsymbol{A}^{\mathrm{c}}_2, \tag{3.23}$$

$$\boldsymbol{G}^{\mathrm{c}}_{123} = -\frac{\delta \Gamma_1}{\delta V_{213}} - \boldsymbol{S}^{\mathrm{c}}_{12} \boldsymbol{A}^{\mathrm{c}}_3. \tag{3.24}$$

From (3.23) we read off that, for instance, cutting a photon line of the one-particle irreducible vacuum diagrams in all possible ways leads to the diagrams contributing to the connected photon two-point function $\boldsymbol{D}^{\mathrm{c}}_{12}$.

The connected n-point functions are related to the one-particle irreducible n-point functions. The electron and photon self-energy, which we shall denote by Σ^e and Σ^γ, is defined according to

$$\Sigma^e_{12} \equiv S^{-1}_{12} - \boldsymbol{S}^{\mathrm{c}-1}_{12}, \tag{3.25}$$

$$\Sigma^\gamma_{12} \equiv D^{-1}_{12} - \boldsymbol{D}^{\mathrm{c}-1}_{12}, \tag{3.26}$$

where $\boldsymbol{S}^{\mathrm{c}-1}_{12}$ and $\boldsymbol{D}^{\mathrm{c}-1}_{12}$ represent the inverse of the propagators $\boldsymbol{S}^{\mathrm{c}}_{12}$ and $\boldsymbol{D}^{\mathrm{c}}_{12}$, respectively:

$$\int_2 \boldsymbol{S}^{\mathrm{c}}_{12} \boldsymbol{S}^{\mathrm{c}-1}_{23} = \delta_{13}, \tag{3.27}$$

$$\int_2 \boldsymbol{D}^{\mathrm{c}}_{12} \boldsymbol{D}^{\mathrm{c}-1}_{23} = \delta_{13}. \tag{3.28}$$

Due to (3.25) and (3.26) the connected electron and photon two-point functions follow from the

Dyson equations

$$\boldsymbol{S}^{\rm c}_{12} = S_{12} + \int_{34} S_{13}\Sigma^{e}_{34}\boldsymbol{S}^{\rm c}_{42}, \tag{3.29}$$

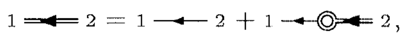
$$\boldsymbol{D}^{\rm c}_{12} = D_{12} + \int_{34} D_{13}\Sigma^{\gamma}_{34}\boldsymbol{D}^{\rm c}_{42}. \tag{3.30}$$

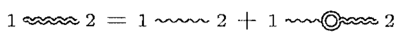
Representing the self-energies Σ^e and Σ^γ by a two-vertex with a big open dot

 $\equiv \Sigma^{e}_{12}$, (3.31)

 $\equiv \Sigma^{\gamma}_{12}$, (3.32)

the Dyson equations (3.29) and (3.30) read graphically

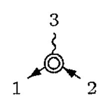
 (3.33)

 (3.34)

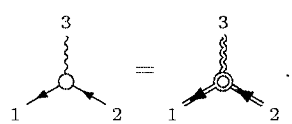
The one-particle irreducible three-point function τ is defined by

$$\boldsymbol{G}^{\rm c}_{123} = -\int_{456} \boldsymbol{S}^{\rm c}_{14}\boldsymbol{S}^{\rm c}_{52}\boldsymbol{D}^{\rm c}_{36}\tau_{456}. \tag{3.35}$$

Representing the one-particle irreducible three-point function τ by a three-vertex with a big open dot

 $\equiv -\tau_{123}$, (3.36)

relation (3.35) is pictured by

 (3.37)

For later purposes we note that the one-particle three-point function τ may be also defined by

$$\frac{\delta^2\Gamma_1}{\delta S^{-1}_{12}\,\delta \boldsymbol{A}^{\rm c}_{3}} = \int_{45} \boldsymbol{S}^{\rm c}_{24}\boldsymbol{S}^{\rm c}_{51}\tau_{453}, \tag{3.38}$$

as follows from (3.20)–(3.22), (3.34), and (3.35).

B. Closed Set of Schwinger–Dyson Equations for One-Particle Irreducible n-Point Functions

Now we calculate the effect of the functional Legendre transform with respect to the current upon the trivial functional identities (2.52), (2.55), and (2.63) which immediately followed from the definition of the functional integral. This leads to a closed set of equations determining the electron and photon self-energy as well as the one-particle irreducible three-point function.

1. *Electron Self-Energy*

In order to determine the electron self-energy Σ^e, we start with the first functional differential equation (2.52) for the vacuum energy W and perform the functional Legendre transform of the first kind defined in Section III.A. Inserting (3.1), (3.13), and (3.19) by taking into account the compatibility relation (3.20), we thus obtain

$$\delta_{12} + \int_3 S^{-1}_{13} \frac{\delta \Gamma_1}{\delta S^{-1}_{23}} + \int_{345} V_{134} \frac{\delta^2 \Gamma_1}{\delta S^{-1}_{23} \delta \boldsymbol{A}^c_5} \left\{ 2 \frac{\delta \Gamma_1}{\delta D^{-1}_{45}} - \boldsymbol{A}^c_4 \boldsymbol{A}^c_5 \right\} + \int_{34} V_{134} \boldsymbol{A}^c_4 \frac{\delta \Gamma_1}{\delta S^{-1}_{23}} = 0. \tag{3.39}$$

With the definition of the connected electron and photon two-point function (3.22) and (3.23), as well as the one-particle irreducible three-point function (3.38), we get

$$\delta_{12} - \int_3 S^{-1}_{13} \boldsymbol{S}^c_{32} + \int_{34567} V_{134} \tau_{567} \boldsymbol{S}^c_{62} \boldsymbol{S}^c_{35} \boldsymbol{D}^c_{47} - \int_{34} V_{134} \boldsymbol{S}^c_{32} \boldsymbol{A}^c_4 = 0. \tag{3.40}$$

This reduces to the Schwinger–Dyson equation for the electron self-energy (3.25)

$$\Sigma^e_{12} = \int_{3456} V_{134} \tau_{526} \boldsymbol{S}^c_{35} \boldsymbol{D}^c_{46} - \int_3 V_{123} \boldsymbol{A}^c_3, \tag{3.41}$$

which graphically reads

(3.42)

2. *Photon Self-Energy*

The photon self-energy Σ^γ follows in the same way from the third functional identity (2.63). We perform the functional Legendre transform of the first kind by using (3.1), (3.13), (3.14), (3.19), and the compatibility relation (3.20):

$$\delta_{12} + \frac{\delta \Gamma_1}{\delta \boldsymbol{A}^c_1} \boldsymbol{A}^c_2 - 2 \int_3 D^{-1}_{13} \frac{\delta \Gamma_1}{\delta D^{-1}_{23}} - \int_{345} V_{341} \frac{\delta^2 \Gamma_1}{\delta S^{-1}_{34} \delta \boldsymbol{A}^c_5} \left\{ 2 \frac{\delta \Gamma_1}{\delta D^{-1}_{25}} - \boldsymbol{A}^c_2 \boldsymbol{A}^c_5 \right\} - \int_{34} V_{341} \frac{\delta \Gamma_1}{\delta S^{-1}_{34}} \boldsymbol{A}^c_2 = 0. \tag{3.43}$$

The functional derivative with respect to the field expectation value $\boldsymbol{A}^c$ in the second term can be eliminated by performing the functional Legendre transform of the first kind in the second functional identity (2.55). With (3.1), (3.5), and (3.13) we obtain

$$\frac{\delta \Gamma_1}{\delta \boldsymbol{A}^c_1} = \int_2 D^{-1}_{12} \boldsymbol{A}^c_2 + \int_{23} V_{231} \frac{\delta \Gamma_1}{\delta S^{-1}_{23}}. \tag{3.44}$$

Inserting this into (3.43) leads to

$$\delta_{12} - \int_3 D^{-1}_{13} \left\{ 2 \frac{\delta \Gamma_1}{\delta D^{-1}_{23}} - \boldsymbol{A}^c_2 \boldsymbol{A}^c_3 \right\} - \int_{345} V_{341} \frac{\delta^2 \Gamma_1}{\delta S^{-1}_{34} \delta \boldsymbol{A}^c_5} \left\{ 2 \frac{\delta \Gamma_1}{\delta D^{-1}_{25}} - \boldsymbol{A}^c_2 \boldsymbol{A}^c_5 \right\} = 0. \tag{3.45}$$

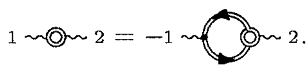
Using once more the relations (3.23) and (3.38), we find

$$\delta_{12} - \int_3 D^{-1}_{13} \boldsymbol{D}^{\mathrm{c}}_{23} - \int_{34567} V_{341} \tau_{675} \boldsymbol{S}^{\mathrm{c}}_{46} \boldsymbol{S}^{\mathrm{c}}_{73} \boldsymbol{D}^{\mathrm{c}}_{25} = 0. \tag{3.46}$$

Thus the Schwinger–Dyson equation for the photon self-energy (3.26) reads

$$\Sigma^{\gamma}_{12} = - \int_{3456} V_{341} \tau_{562} \boldsymbol{S}^{\mathrm{c}}_{63} \boldsymbol{S}^{\mathrm{c}}_{45}, \tag{3.47}$$

with the graphical representation



(3.48)

3. *One-Particle Irreducible Three-Point Function*

The iteration of the integral equations (3.42) and (3.48) for the electron and photon self-energy Σ^e and Σ^γ requires the knowledge of the one-particle irreducible three-point function τ. To obtain this, we further evaluate (3.38) by inserting (3.44). Thus the one-particle irreducible three-point function τ follows from

$$\tau_{123} = \int_{4567} \boldsymbol{S}^{\mathrm{c}-1}_{14} \boldsymbol{S}^{\mathrm{c}-1}_{52} \frac{\delta^2 \Gamma_1}{\delta S^{-1}_{54} \delta S^{-1}_{67}} V_{673}. \tag{3.49}$$

To express the right-hand side in terms of the electron self-energy, we apply a functional derivative with respect to the electron kernel S^{-1}_{45} to the identity

$$\int_2 \boldsymbol{S}^{\mathrm{c}}_{12} \boldsymbol{S}^{\mathrm{c}-1}_{23} = \delta_{13}, \tag{3.50}$$

yielding

$$\frac{\delta \boldsymbol{S}^{\mathrm{c}-1}_{12}}{\delta S^{-1}_{45}} = - \int_{67} \boldsymbol{S}^{\mathrm{c}-1}_{17} \boldsymbol{S}^{\mathrm{c}-1}_{62} \frac{\delta \boldsymbol{S}^{\mathrm{c}}_{76}}{\delta S^{-1}_{45}}, \tag{3.51}$$

so that we obtain together with (3.22)

$$\frac{\delta \boldsymbol{S}^{\mathrm{c}-1}_{12}}{\delta S^{-1}_{45}} = \int_{67} \boldsymbol{S}^{\mathrm{c}-1}_{17} \boldsymbol{S}^{\mathrm{c}-1}_{62} \frac{\delta^2 \Gamma_1}{\delta S^{-1}_{45} \delta S^{-1}_{67}}. \tag{3.52}$$

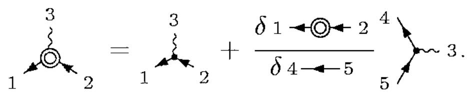
From (3.49) and (3.52) we conclude

$$\tau_{123} = \int_{45} \frac{\delta \boldsymbol{S}^{\mathrm{c}-1}_{12}}{\delta S^{-1}_{45}} V_{453}. \tag{3.53}$$

Inserting (3.25) and using the functional chain rule (2.68), we finally arrive at a functional integrodifferential equation for the one-particle irreducible three-point function

$$\tau_{123} = V_{123} + \int_{4567} V_{453} S_{64} S_{57} \frac{\delta \Sigma^e_{12}}{\delta S_{67}}, \tag{3.54}$$

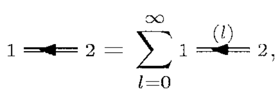
whose graphical representation is

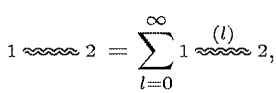

(3.55)

Thus the diagrams of the one-particle irreducible three-point function follow from those of the electron self-energy by inserting a three-vertex in an electron line in all possible ways. Note that the graphical content (3.55) of the functional integrodifferential equation (3.54) can be heuristically deduced from the local current conservation law of QED and its corresponding Ward identity (see, for example, the detailed discussion in Ref. [18]).

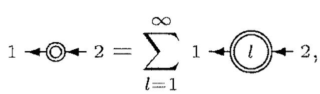
C. Graphical Recursion Relations

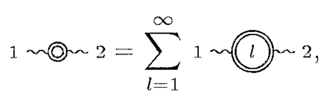
We now demonstrate how the diagrams of the connected electron and photon two-point function, the electron and photon self-energy, as well as the one-particle irreducible three-point function are recursively determined in a graphical way, generating all one-particle irreducible Feynman diagrams which are needed for the renormalization of QED. To simplify the discussion, we restrict ourselves to a vanishing field expectation value, so that we can neglect the last term in Eq. (3.42). Performing a loop expansion of the connected electron and photon two-point function


(3.56)

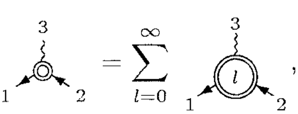

(3.57)

of their corresponding self-energies

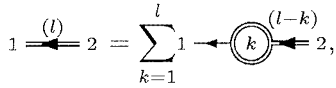

(3.58)


(3.59)

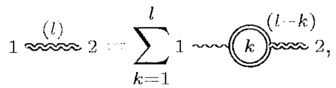
as well as of the one-particle irreducible three-point function


(3.60)

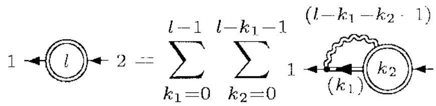
we obtain from (3.33), (3.34), (3.42), (3.48), and (3.55) the following closed set of graphical recursion relations:



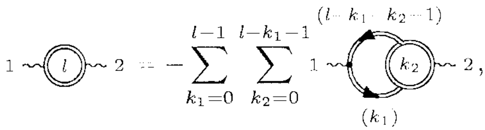
(3.61)



(3.62)

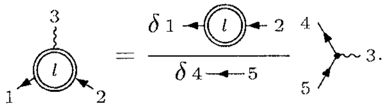


(3.63)



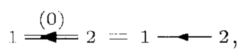
(3.64)

and

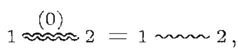


(3.65)

These are solved starting from the zeroth-loop contribution to the connected electron and photon two-point function

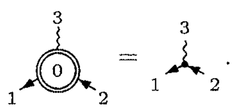


(3.66)



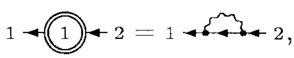
(3.67)

and the one-particle irreducible three-point function

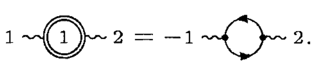


(3.68)

First, we insert (3.66)–(3.68) into (3.63) and (3.64) to obtain the one-loop contribution to the electron and the photon self-energy:

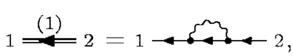


(3.69)

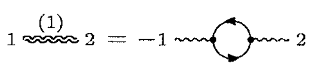


(3.70)

With this we find from (3.61) and (3.62) the corresponding connected two-point functions in the one-loop order:

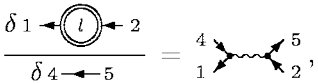


(3.71)



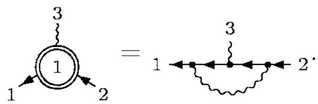
(3.72)

Amputating one electron line from (3.69),



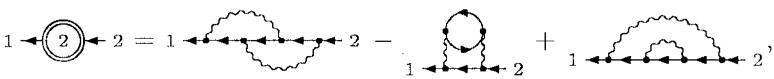
(3.73)

we determine from (3.65) the one-loop contribution to the one-particle irreducible three-point function

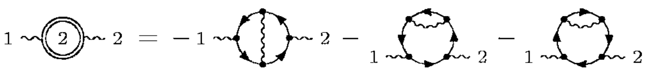


(3.74)

Using Eqs. (3.63) and (3.64), we then find the electron and photon self-energy with two loops:

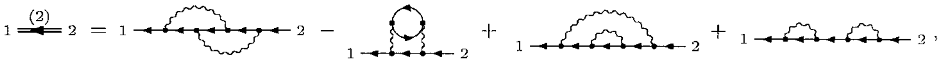


(3.75)

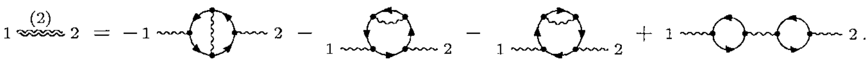


(3.76)

The corresponding contributions to the connected two-point functions are according to (3.61) and (3.62):

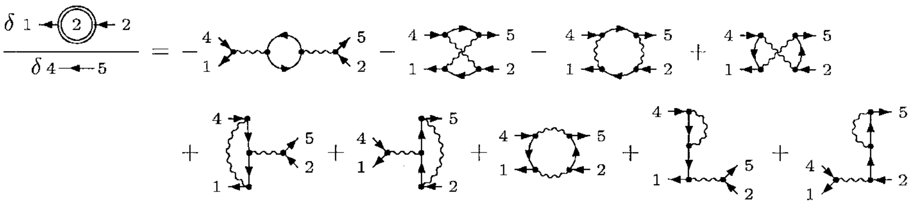


(3.77)



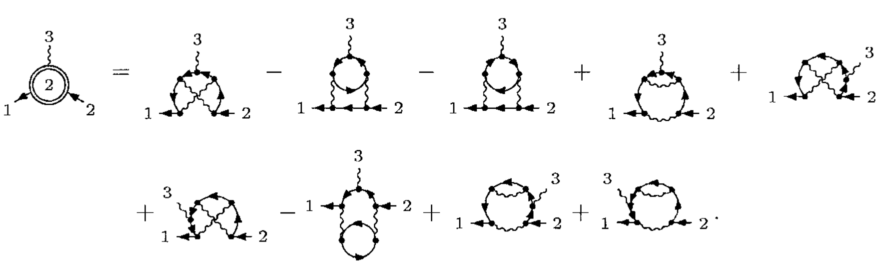
(3.78)

Comparing (2.85) and (2.86) with (3.77) and (3.78) shows the graphical consequence of the functional Legendre transform with respect to the current by the example of the connected two-point functions. It systematically eliminates the one-particle reducible diagrams carrying any kind of tadpole correction. The subsequent amputation of one electron line in (3.75) leads to



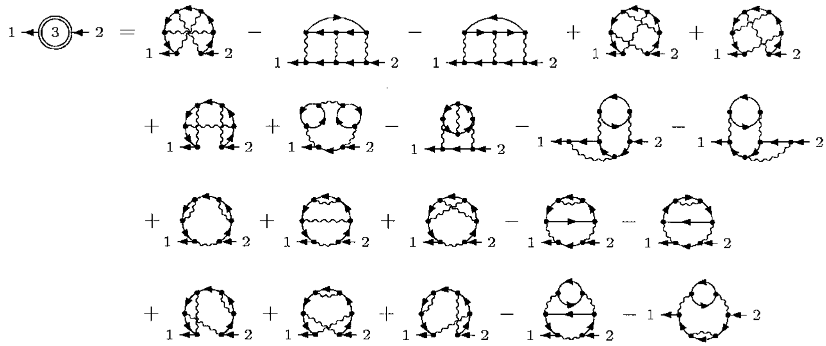
(3.79)

so we find from (3.65) the one-particle irreducible three-point function with two loops:



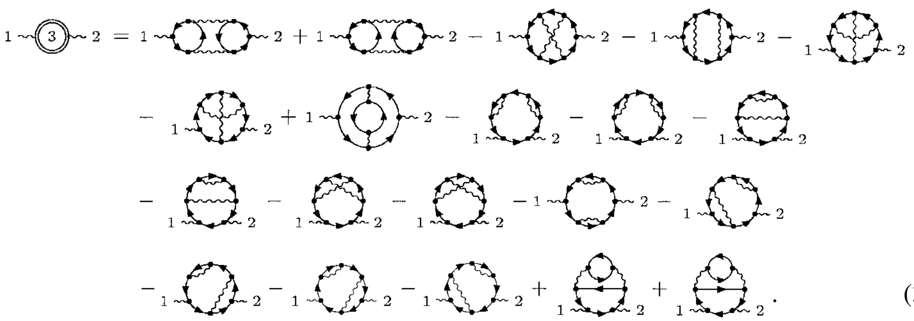
(3.80)

As a consequence, we obtain from (3.63) and (3.64) the three-loop contribution of the electron self-energy



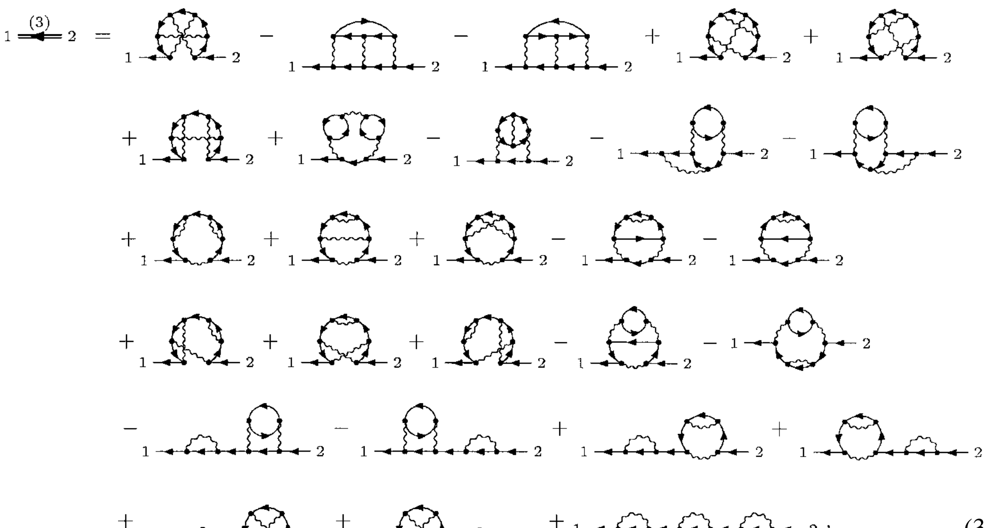
(3.81)

and of the photon self-energy

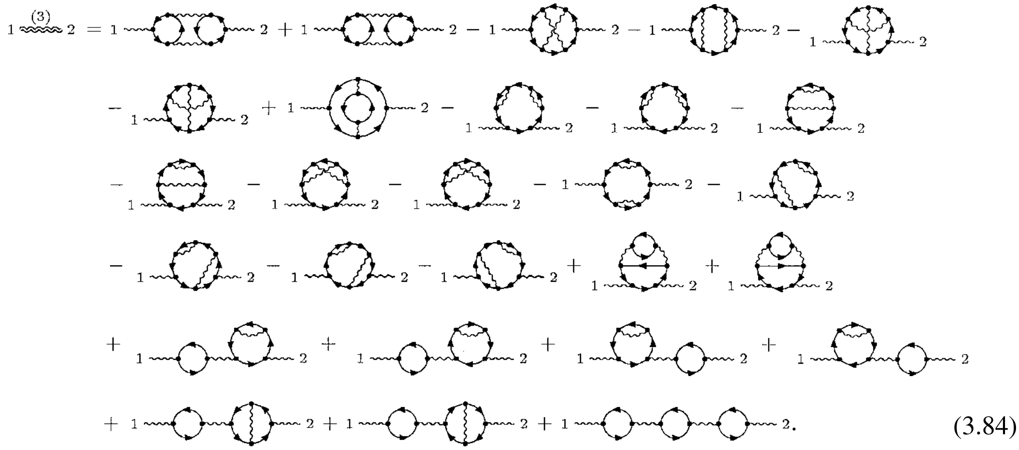


(3.82)

With this, the corresponding connected two-point functions for three loops follow from (3.61) and (3.62):



(3.83)



(3.84)

D. One-Particle Irreducible Vacuum Diagrams

By analogy with the discussion of the connected vacuum diagrams in Section II.D, also the one-particle irreducible vacuum diagrams of QED can be generated together with their weights in two different ways. First, we show that they can be constructed from the diagrams of the electron and photon connected two-point function as well as the one-particle irreducible three-point function which have already been determined in the previous subsection. Second, we derive from the functional identities developed so far a nonlinear functional differential equation for the effective energy of the first kind and convert it into a graphical recursion relation which directly generates the one-particle irreducible vacuum diagrams.

1. *Relation to the Diagrams of the One-Particle Irreducible n-Point Functions*

After having iteratively solved the closed set of graphical recursion relations (3.61)–(3.65) for the electron and photon self-energy and the connected two-point function as well as the one-particle irreducible three-point function, the corresponding loopwise one-particle irreducible vacuum diagrams can be constructed as follows. Going back to the defining equations (3.22)–(3.24), we obtain with (3.35), with the functional chain rule (2.68) and its analogue

$$\frac{\delta}{\delta D_{12}^{-1}} = -\frac{1}{2}\int_{34}\{D_{13}D_{24} + D_{14}D_{23}\}\frac{\delta}{\delta D_{34}}, \tag{3.85}$$

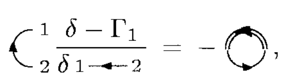
three functional differential equations for the effective energy of the first kind:

$$\int_{12} S_{12}\frac{\delta\Gamma_1}{\delta S_{12}} = \int_{12} S_{21}^{-1}\boldsymbol{S}_{12}^{\mathrm{c}}, \tag{3.86}$$

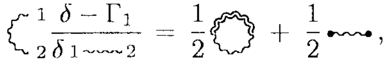
$$\int_{12} D_{12}\frac{\delta\Gamma_1}{\delta D_{12}} = -\frac{1}{2}\int_{12} D_{12}^{-1}\boldsymbol{D}_{12}^{\mathrm{c}} - \frac{1}{2}\int_{12} D_{12}^{-1}\boldsymbol{A}_1^{\mathrm{c}}\boldsymbol{A}_2^{\mathrm{c}}, \tag{3.87}$$

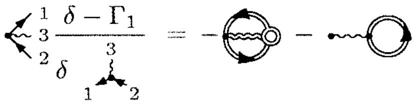
$$\int_{123} V_{123}\frac{\delta\Gamma_1}{\delta V_{123}} = \int_{123456} V_{123}\tau_{456}\boldsymbol{S}_{24}^{\mathrm{c}}\boldsymbol{S}_{51}^{\mathrm{c}}\boldsymbol{D}_{36}^{\mathrm{c}} - \int_{123} V_{123}\boldsymbol{S}_{21}^{\mathrm{c}}\boldsymbol{A}_3^{\mathrm{c}}. \tag{3.88}$$

Their graphical representiations are



(3.89)


(3.90)


(3.91)

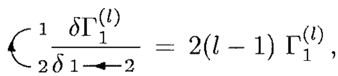
They are based on counting the electron lines, the photon lines, and the three-vertices of the one-particle irreducible vacuum diagrams. If the interaction V vanishes, Eqs. (3.89)–(3.91) are solved by the free effective energy of the first kind (3.10) due to (3.66)–(3.68). For a non-vanishing interaction V Eqs. (3.89)–(3.91) produce corrections to (3.10) which we shall denote as $\Gamma_1^{(\mathrm{int})}$. Thus the effective energy of the first kind Γ_1 decomposes according to

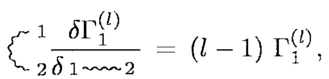
$$\Gamma_1 = \Gamma_1^{(\mathrm{free})} + \Gamma_1^{(\mathrm{int})}. \tag{3.92}$$

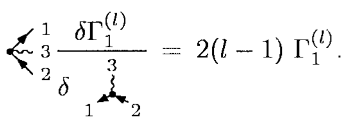
In the following we recursively determine $\Gamma_1^{(\mathrm{int})}$ in a graphical way for a vanishing field expectation value, so that we can neglect the last term in both (3.90) and (3.91). Performing a loopwise expansion of the interaction part of the effective energy of the first kind,

$$\Gamma_1^{(\mathrm{int})} = \sum_{l=2}^{\infty} \Gamma_1^{(l)}, \tag{3.93}$$

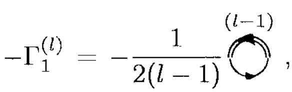
we use the following eigenvalue problems for $l \geq 2$:

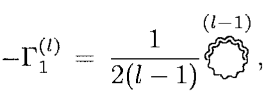

(3.94)

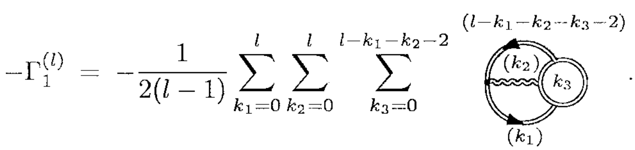

(3.95)


(3.96)

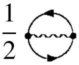
With these we explicitly solve (3.89)–(3.91) for the expansion coefficients $\Gamma_1^{(l)}$ and obtain for $l \geq 2$


(3.97)

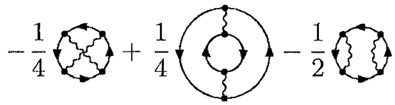

(3.98)


(3.99)

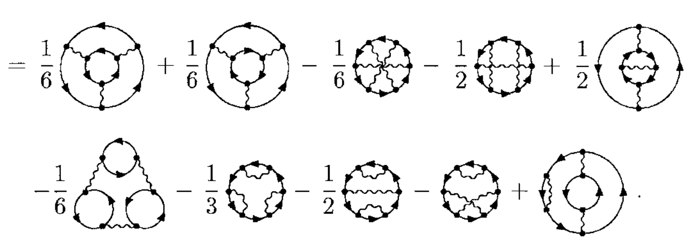
Inserting (3.66)–(3.84) for the lower loop contributions to the connected electron and photon two-point function as well as the one-particle irreducible three-point function in one of Eqs. (3.97)–(3.99), we get the effective energy of the first kind for $l = 2$ loops,

$$-\Gamma_1^{(2)} = -\frac{1}{2}$$  , (3.100)

for $l = 3$ loops

$$-\Gamma_1^{(3)} =$$  , (3.101)

and for four loops

$$-\Gamma_1^{(4)} =$$  . (3.102)

A comparison with the corresponding results for the vacuum energy in (2.102), (2.103), and (2.110) shows that the effective energy of the first kind contains precisely all one-particle irreducible vacuum diagrams.

2. *Graphical Recursion Relation*

Each of the three functional differential equations for the effective energy of the first kind (3.86)–(3.88) can be used to derive a graphical recursion relation which directly generates the one-particle irreducible vacuum diagrams. Here we restrict ourselves to the functional differential equation (3.86) which is based on counting the number of electron lines of the one-particle irreducible vacuum diagrams. Inserting (3.22), (3.23), (3.25), (3.41), and (3.49) for the connected electron and photon two-point function, the electron self-energy, and the one-particle irreducible three-point function, we obtain from (3.86):

$$\delta_{11}\int_1 + \int_{12} S_{12}^{-1}\frac{\delta\Gamma_1}{\delta S_{12}^{-1}} = -\int_{123456} V_{123}V_{456}\frac{\delta^2\Gamma_1}{\delta S_{12}^{-1}\delta S_{45}^{-1}}\left\{2\frac{\delta\Gamma_1}{\delta D_{36}^{-1}} - A_3^{\mathrm{c}}A_6^{\mathrm{c}}\right\} - \int_{123} V_{123}\frac{\delta\Gamma_1}{\delta S_{12}^{-1}}A_3^{\mathrm{c}}. \tag{3.103}$$

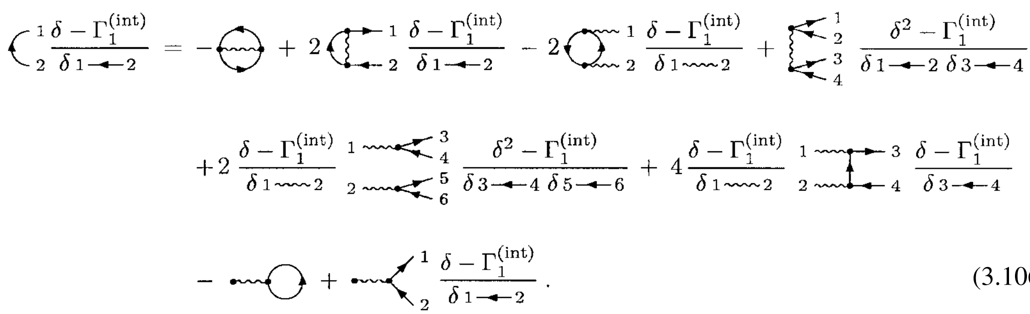
If the interaction V vanishes, this is solved by the free effective energy of the first kind (3.9), which has the functional derivatives

$$\frac{\delta\Gamma_1^{(\mathrm{free})}}{\delta D_{12}^{-1}} = \frac{1}{2}D_{12} + \frac{1}{2}A_1^{\mathrm{c}}A_2^{\mathrm{c}}, \qquad \frac{\delta\Gamma_1^{(\mathrm{free})}}{\delta S_{12}^{-1}} = -S_{21}, \qquad \frac{\delta^2\Gamma_1^{(\mathrm{free})}}{\delta S_{12}^{-1}\delta S_{34}^{-1}} = S_{23}S_{41}. \tag{3.104}$$

For a non-vanishing interaction V, the right-hand side in (3.103) corrects (3.9) by the interaction part of the effective energy of the first kind $\Gamma_1^{(\mathrm{int})}$. Inserting the decomposition (3.92) into (3.103) and using (3.104), we obtain together with the functional chain rule the following functional differential equation for the interaction part of the effective energy of the first kind $\Gamma_1^{(\text{int})}$:

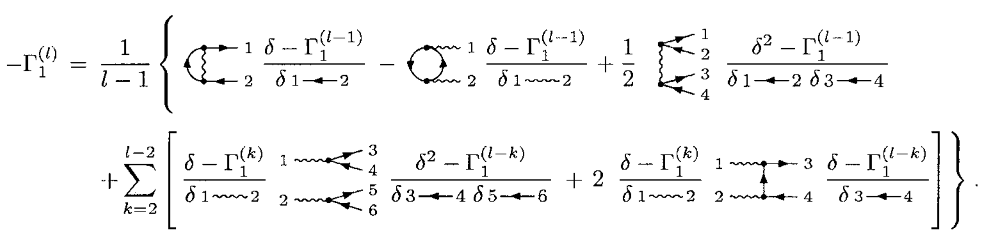
$$
\begin{aligned}
\int_{12} S_{12} \frac{\delta \Gamma_1^{(\text{int})}}{\delta S_{12}} = & \int_{123456} V_{123} V_{456} S_{24} S_{51} D_{36} + 2 \int_{12345678} V_{123} V_{456} D_{36} S_{51} S_{28} S_{74} \frac{\delta \Gamma_1^{(\text{int})}}{\delta S_{78}} \\
& - 2 \int_{12345678} V_{123} V_{456} S_{24} S_{51} D_{37} D_{68} \frac{\delta \Gamma_1^{(\text{int})}}{\delta D_{78}} \\
& + \int_{123456789\bar{1}} V_{123} V_{456} D_{36} S_{71} S_{28} S_{94} S_{5\bar{1}} \frac{\delta^2 \Gamma_1^{(\text{int})}}{\delta S_{78} \delta S_{9\bar{1}}} \\
& - 2 \int_{123456789\bar{1}\bar{2}\bar{3}} V_{123} V_{456} S_{71} S_{28} S_{94} S_{5\bar{1}} D_{3\bar{2}} D_{6\bar{3}} \frac{\delta^2 \Gamma_1^{(\text{int})}}{\delta S_{78} \delta S_{9\bar{1}}} \frac{\delta \Gamma_1^{(\text{int})}}{\delta D_{\bar{2}\bar{3}}} \\
& - 4 \int_{123456789\bar{1}} V_{123} V_{456} S_{51} S_{74} S_{28} D_{39} D_{6\bar{1}} \frac{\delta \Gamma_1^{(\text{int})}}{\delta S_{78}} \frac{\delta \Gamma_1^{(\text{int})}}{\delta D_{9\bar{1}}} \\
& - \int_{123} V_{123} S_{21} A_3^{\mathrm{c}} - \int_{12345} V_{123} S_{41} S_{25} A_3^{\mathrm{c}} \frac{\delta \Gamma_1^{(\text{int})}}{\delta S_{45}} .
\end{aligned}
\tag{3.105}
$$

Its graphical representation reads



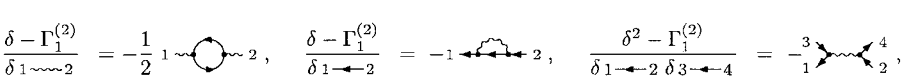
(3.106)

Again, we illustrate the graphical recursive solution of (3.106) only for a vanishing field expectation value, so that we can drop the last two terms. Inserting the loop expansion (3.93) and taking into account the eigenvalue problem (3.94), we obtain a graphical recursion relation for the expansion coefficients $\Gamma_1^{(l)}$ of the effective energy of the first kind for $l \geq 3$:



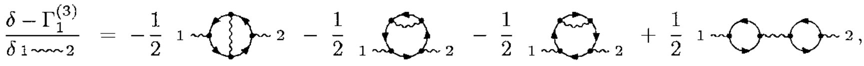
(3.107)

It is solved starting from $\Gamma_1^{(2)}$ in (3.100). With the line amputations



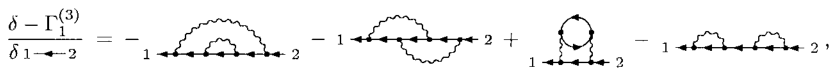
(3.108)

we obtain from (3.107) the three-loop result $\Gamma_1^{(3)}$ in (3.101). Performing the amputation of one photon line



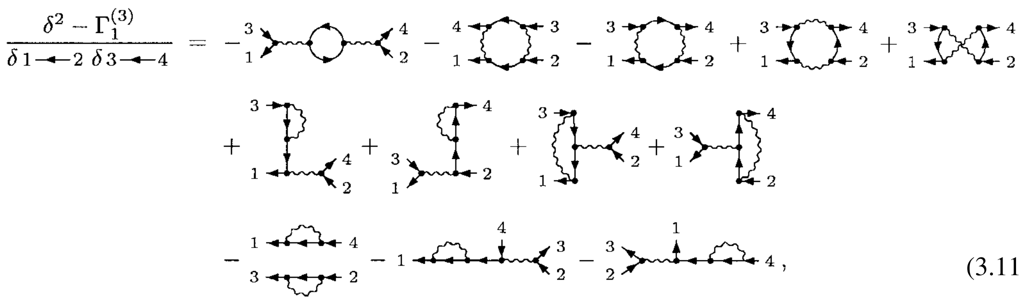
(3.109)

one electrone line



(3.110)

and two electron lines



(3.111)

we obtain from (3.107) the four-loop result $\Gamma_1^{(4)}$ in (3.102).

IV. SUMMARY AND OUTLOOK

We have derived a closed set of Schwinger–Dyson equations in QED by using functional analytic methods developed in Refs. [8, 9]. Their conversion to graphical recursion relations allows us to systematically generate connected and one-particle irreducible Feynman diagrams for n-point functions. In the subsequent paper [27] we show that corrections of the electron and the photon propogator as well as the vertex can be iteratively eliminated by introducing higher Legendre transformations [8, 9]. This will lead to graphical recursion relations for all skeleton Feynman diagrams in QED [18].

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